$u, v \in \mathbb{R}^n$

$u \cdot v$

$\langle u, v\rangle = \langle v, u\rangle$

$|u\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\quad |v\rangle = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$\langle u|\,|v\rangle$

Inner product and length of vectors

- Define the inner product of $|u\rangle, |v\rangle \in \mathbb{C}^n$ by $\boxed{\langle u|v\rangle = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n = u_1^* v_1 + \cdots + u_n^* v_n} \in \mathbb{C}$.
- Then $\langle u|v\rangle = \langle v|u\rangle^*$, $\langle u|c_1 v + c_2 w\rangle = c_1\langle u|v\rangle + c_2\langle u|w\rangle$ for any $c_1, c_2 \in \mathbb{C}$, $|u\rangle, |v\rangle, |w\rangle \in \mathbb{C}^n$.
  $= \overline{\langle v|u\rangle}$
- Also, $\langle u, u\rangle = \sum_{j=1}^n |u_j|^2 \geq 0$, where the equality holds if and only if $|u\rangle = |0\rangle$.
- Define $\|u\| = \sqrt{\langle u|u\rangle}$ as the length or norm of $|u\rangle$.
- Two vectors $|u\rangle, |v\rangle \in \mathbb{C}^n$ are orthogonal if $\langle u, v\rangle = 0$.

Example $\quad |u\rangle = \begin{bmatrix} 1+i \\ 2-i \end{bmatrix} \quad |v\rangle = \begin{bmatrix} 3+2i \\ 1+6i \end{bmatrix}$

$$\langle u|v\rangle = (1-i,\ 2+i)\begin{pmatrix} 3+2i \\ 1+6i \end{pmatrix} = (1-i)(3+2i) + (2+i)(1+6i)$$
$$= (3+2) - i + (2-6) + 13i$$
$$= 1+12i$$

$$\langle u|v\rangle = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n$$
$$\langle v|u\rangle = \bar{v}_1 u_1 + \cdots + \bar{v}_n u_n \quad \therefore \langle v|u\rangle^*$$
$$= \overline{\bar{v}_1 u_1 + \cdots + \bar{v}_n u_n}$$
$$= \overline{\bar{v}_1 u_1} + \cdots + \overline{\bar{v}_n u_n}$$
$$= v_1 \bar{u}_1 + \cdots + v_n \bar{u}_n = \langle u|v\rangle$$

Warning: In mathematics $u, v \in \mathbb{C}^n$

$$\langle u, v\rangle = \sum_{i=1}^n \bar{v}_i u_i$$

so that
$$\langle \alpha u_1 + \beta u_2, v\rangle = \alpha\langle u_1, v\rangle + \beta\langle u_2, v\rangle$$

$$\langle u, \alpha_1 v_1 + \alpha_2 v_2\rangle = \bar{\alpha}_1\langle u, v_1\rangle + \bar{\alpha}_2\langle u, v_2\rangle$$

In physics,
$$\langle u|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1\langle u|v_1\rangle + \alpha_2\langle u|v_2\rangle$$

$$\langle \alpha_1 u_1 + \alpha_2 u_2|v\rangle$$
$$= (\langle v|\alpha_1 u_1 + \alpha_2 u_2\rangle)^*$$
$$= [\alpha_1\langle v|u_1\rangle + \alpha_2\langle v|u_2\rangle]^*$$
$$= \bar{\alpha}_1\langle v|u_1\rangle^* + \bar{\alpha}_2\langle v|u_2\rangle^*$$
$$= \bar{\alpha}_1\langle u_1|v\rangle + \bar{\alpha}_2\langle u_2|v\rangle$$

Define $\||u\rangle\| = \sqrt{\langle u|u\rangle}$.

Then ① $\||u\rangle\| \geq 0$, with equality if and only if $|u\rangle = |0\rangle$

② $\||u\rangle + |v\rangle\| \leq \||u\rangle\| + \||v\rangle\| \quad \forall |u\rangle, |v\rangle \in \mathbb{C}^n$

③ $\|\alpha|u\rangle\| = |\alpha|\,\||u\rangle\| \quad \forall \alpha \in \mathbb{C}, |v\rangle \in \mathbb{C}^n$

Example $|e_1\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, $|e_2\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ are orthogonal.

$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $|u_2\rangle = \frac{\begin{bmatrix}i\\-i\end{bmatrix}}{\sqrt{2}}$, $\langle u_1|u_2\rangle = \frac{[1,1]}{\sqrt{2}}\frac{\begin{bmatrix}i\\-i\end{bmatrix}}{\sqrt{2}} = \frac{1}{2}(i-i) = 0$

§1.4 Orthonormal basis and the Gram-Schmidt process. $\therefore$ $|u_1\rangle$, $|u_2\rangle$ are orthogonal.

- Express a vector as a linear combination of orthonormal basis $\{|e_1\rangle, \ldots, |e_n\rangle\}$.
- The set $\{P_j = |e_j\rangle\langle e_j| : j = 1, \ldots, n\}$ form a complete set of projection operators/matrices.
- Gram-Schmidt orthogonalization/orthonormalization.

Recall, every $|v\rangle \in \mathbb{C}^n$ is a linear combination of $|u_1\rangle \cdots |u_n\rangle$ if $\{|u_1\rangle, \ldots, |u_n\rangle\}$.

and the linear coefficient $a_1, \ldots, a_n$ are uniquely determined by solving

$$A\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} = \begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix},$$ where the columns of A are $|u_1\rangle \cdots |u_n\rangle$,

because $$\begin{bmatrix}|u_1\rangle \cdots |u_n\rangle\end{bmatrix}\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} = a_1|u_1\rangle + \cdots + a_n|u_n\rangle$$

$$A\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} = \begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix} = |v\rangle$$

Note: If $\{|u_1\rangle \cdots |u_n\rangle\}$ is a orthonormal basis.

i.e., $\langle u_i|u_j\rangle = 0$ if $i \neq j$, $\langle u_i, u_i\rangle = 1$ $\forall i$.

& $\mathcal{U} = [|u_1\rangle \cdots |u_n\rangle]$. $\Rightarrow$ $\mathcal{U}^\dagger \mathcal{U} = \begin{bmatrix}\langle u_1|\\ \vdots\\ \langle u_n|\end{bmatrix}\begin{bmatrix}|u_1\rangle \cdots |u_n\rangle\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ & & \ddots & & 0\\ 0 & & & & 1\end{bmatrix} = I.$

$\mathcal{U}^\dagger = \begin{bmatrix}\overline{u_{ij}}\end{bmatrix}^t$

$\therefore$ $$\mathcal{U}\begin{bmatrix}a_1\\ a_n\end{bmatrix} = |v\rangle$$

$$\Leftrightarrow \mathcal{U}^\dagger\mathcal{U}\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} = \mathcal{U}^\dagger|v\rangle$$

$\therefore$ $$\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} = \mathcal{U}^\dagger|v\rangle = \begin{bmatrix}\langle u_1|v\rangle\\ \vdots\\ \langle u_n|v\rangle\end{bmatrix}$$

Example Let $\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}i\\-i\end{bmatrix}\right\}$. $|v\rangle = \begin{pmatrix}3+2i\\1+6i\end{pmatrix}$

$= \{|u_1\rangle, |u_2\rangle\}$

Then $|v\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle$ with $\alpha_1 = \frac{(1,1)}{\sqrt{2}}\begin{pmatrix}3+2i\\1+6i\end{pmatrix} = \frac{1}{\sqrt{2}}(4+8i)$

$\alpha_2 = \frac{1}{\sqrt{2}}(-i, i)\begin{bmatrix}3+2i\\1+6i\end{bmatrix} = \frac{1}{\sqrt{2}}(-4-2i)$

Suppose $\{|u_1\rangle, \cdots |u_n\rangle\}$ is an orthonormal basis for $\mathbb{C}^n$.

Then $\quad \mathcal{U}^+ \mathcal{U} = I_n \quad$ if $\quad \mathcal{U} = [|u_1\rangle, \cdots, |u_n\rangle]$.

$(u_{ij})^+ \quad (u_{ij})$

So $\quad \mathcal{U}\mathcal{U}^+ = I_n = [|u_1\rangle \cdots |u_n\rangle] \begin{bmatrix} \langle u_1| \\ \vdots \\ \langle u_n| \end{bmatrix}$

$$= |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| + \cdots + |u_n\rangle\langle u_n|$$

$$= P_1 + P_2 + \cdots + P_n \quad \cancel{\text{s.t.}}.$$

s.t. $\quad P_i = P_i^+ = P_i^2 \quad$ and $\quad P_1 + \cdots + P_n = I_n \quad$ and $P_i P_j = 0 \quad i \neq j$

$\therefore \{P_1 \cdots P_n\}$ is a complete set of rank 1. orthogonal projections

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Gram-Schmidt Process for constructing orthonormal basis spanned by $\{|v_1\rangle, \cdots, |v_m\rangle\}$, ~~where $|v\rangle$~~ a linearly independent set.

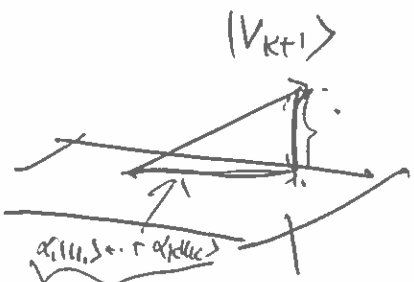
as follows.

Let $\quad |u_1\rangle = \dfrac{|v_1\rangle}{\| v_1 \|}$.

Suppose $\{|u_1\rangle, \cdots, |u_k\rangle\}$ ~~any~~ is an orthonormal set constructed from $\{|v_1\rangle, \cdots |v_k\rangle\}$. $1 \le k < m$

Then construct $\quad |u_{k+1}\rangle = \dfrac{|v_{k+1}\rangle - (\alpha_1 |u_1\rangle + \cdots + \alpha_k |u_k\rangle)}{\| \, |v_{k+1}\rangle - (\alpha_1 |u_1\rangle + \cdots + \alpha_k |u_k\rangle) \|}$

where $\quad \alpha_i = \langle u_i | v_{k+1}\rangle \quad \forall i = 1, \cdots, k$.



Span $\{|u_1\rangle, \cdots, |u_k\rangle\}$

$= $ Span $\{|v_1\rangle, \cdots, |v_k\rangle\}$

<u>Example</u>:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1+i \\ i \\ 0 \end{bmatrix} \right\}$$

$$|u_1\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad |u_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad |u_3\rangle = \frac{1}{\sqrt{12}} \begin{bmatrix} 1-i \\ -2+2i \\ 1-i \end{bmatrix}$$

$1+1+4+4+1+1$

<u>Question</u>:

5×3

Suppose $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\} \subseteq \mathbb{C}^5$.

If we want to use the $\overset{5\times 5}{A} = \overset{5\times 5}{U} \overset{5\times 5}{R} = U \begin{bmatrix} \diagdown \\ 0 \end{bmatrix}$ command in Matlab.

<u>How can we solve this problem ?</u>

Answer find easy $|v_4\rangle, |v_5\rangle$ to form

$A = \begin{bmatrix} |v_1\rangle \cdots |v_5\rangle \end{bmatrix}$ to do UR <u>decomposition</u>