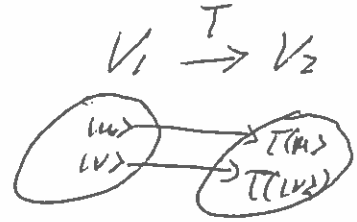


§1.5 Linear operators and Matrices

- A function $T : \mathbb{C}^n \to \mathbb{C}^n$ is linear if $T(|u\rangle + |v\rangle) = T(|u\rangle) + T(|v\rangle)$ and $T(c|u\rangle) = cT(|u\rangle)$ for all $c \in \mathbb{C}$, $|u\rangle, |v\rangle \in \mathbb{C}^n$.
- Every linear function on $\mathbb{C}^n$ has the form $T(|v\rangle) = A|v\rangle$ for an $n \times n$ matrix $A = (A_{ij})$.
- Let A be a matrix. We can define the transpose $\boxed{A^t}$, the conjugate A^* or sometimes $\bar{A}$, and the Hermitian conjugate $A^\dagger = (\bar{A})^t$.
- A square matrix A is Hermitian if $A = A^\dagger$; A is unitary if $\boxed{A^\dagger A = I_n}$; A is normal if $A^\dagger A = AA^\dagger$.

Example (Construction)

$$\mathcal{B} = \{|e_1\rangle, \dots, |e_n\rangle\}$$

$$T(|e_1\rangle) = |a_1\rangle = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{n1} \end{bmatrix}$$

$$\vdots$$

$$T(|e_n\rangle) = |a_n\rangle = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$$

Then

$$T\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}\right) = T(u_1|e_1\rangle + \dots + u_n|e_n\rangle)$$

$$= u_1 T(|e_1\rangle) + \dots + u_n T(|e_n\rangle)$$

$$= u_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + \dots + u_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$$

$|u\rangle$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A(|u\rangle)$$

Given $A = (A_{ij})$, $A^t = (B_{ij})$ s.t. $B_{ij} = A_{ji}$

$\bar{A} = A^* = (B_{ij})$ s.t. $B_{ij} = \overline{A_{ij}}$

$A^\dagger = (\bar{A})^t = (B_{ij})$ s.t. $B_{ij} = \overline{A_{ji}}$
$= \overline{(A^t)}$

$$A = \begin{pmatrix} 1+2i & 3-i \\ 4 & 5+7i \end{pmatrix} \quad A^t = \begin{pmatrix} 1+2i & 4 \\ 3-i & 5+7i \end{pmatrix}$$

$$A^* = \bar{A} = \begin{pmatrix} 1-2i & 3+i \\ 4 & 5-7i \end{pmatrix}$$

$$A^\dagger = \begin{pmatrix} 1-2i & 4 \\ 3+i & 5-7i \end{pmatrix}$$

Example Hermitian

$A = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$ is Hermitain

$$A^{+} = \overline{\left(\quad \right)^{t}} = \overline{\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}} = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} = A$$

$A = A^{+}$ iff $\cancel{A_{ij} = \overline{A_{ji}}}$, $A_{ij} = \overline{A_{ji}}$ $\forall$ i, j

---

$A = \begin{bmatrix} 1+i & x \\ x & x \end{bmatrix}$ is not Hermitain

$A = \begin{bmatrix} 2 & 1+i \\ 1+i & 3 \end{bmatrix}$ is not Hermitian

Observable / Measurement Operator

Example (unitary) A is unitary if $A^{*}A = I_n$, $AA^{+} = I_n$

i.e., columns of A form an orthonormal basis.

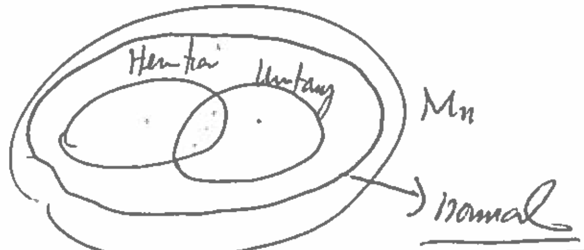
Example (normal) A is normal if $A^{+}A = A^{*}A^{+}$

In particular, if $A = A^{+}$ then $A^{+}A = A^{2} = AA^{+}$

$\therefore$ A is normal.

If A is unitary then $A^{+}A = I_n, \Rightarrow AA^{+} = I_n = A^{+}A$

$\therefore$ A is normal.



Example: $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ unitary $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ is Hermitian

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is Hermitian $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not normal

$AA^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $A^{+}A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Hadamand, Schur, (entrywise)

$\sigma_0 = I_2$

§1.7 Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and their properties

$\delta_{uv} = \begin{cases} 1 & \text{if } u=v \\ 0 & \text{otherwise} \end{cases}$

- The Jordan product $\{\sigma_u, \sigma_v\} = \sigma_u\sigma_v + \sigma_v\sigma_u = 2\delta_{uv}I_2$, where δ_{uv} is the Kronecker symbol.
- The Lie product $[\sigma_u, \sigma_v] = \sigma_u\sigma_v - \sigma_v\sigma_u = 2i\xi_{uv}\sigma_w$ where $\xi_{uv} = 1$ if $(u, v) = (x, y), (y, z), (z, x)$ and $\xi_{uv} = -1$ if $(u, v) = (y, x), (z, y), (x, z)$.

Example

$$\sigma_x\sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$+)\quad \sigma_y\sigma_x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \Bigg(+ \Bigg\} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\sigma_x\sigma_x + \sigma_x\sigma_x = 2\sigma_x^2 = 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2I_2.$$

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma_z$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_y\sigma_x - \sigma_x\sigma_y = -2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -2i\sigma_z$$

$|x+iy| = \sqrt{x^2+y^2}$

§1.6 & 1.8 Eigenvalues

Let A be an $n \times n$ complex matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if there is a nonzero eigenvector $|\lambda\rangle$ such that $A|\lambda\rangle = \lambda|\lambda\rangle$.

- The matrix A always has complex eigenvalues because $\det(tI - A) = 0$ always has a solution λ so that $(\lambda I - A)|\lambda\rangle = |0\rangle$ has non-trivial solution.
- There is a unitary matrix U such that $U^\dagger AU = T$ is in upper triangular form. Moreover, $\det(tI - A) = \det(tI - T) = (t - T_{11}) \cdots (t - T_{nn})$. Schur Triangularization Lemma $T = \begin{bmatrix} T_{11} & * \\ 0 & T_{nn} \end{bmatrix}$
- The matrix A is normal if and only if T is diagonal; the matrix A is unitary if and only if T is a diagonal matrix so that the diagonal entries have moduli 1; the matrix A is Hermitian if and only if T is a real diagonal matrices.

Example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda = 0$ $|\lambda\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that $A|\lambda\rangle = |0\rangle = 0|\lambda\rangle$

Example $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\det(tI-A) = t^2+1 = 0$ has no real solution. $\therefore$ no real values.

Over complex $t^2+1 = (t+i)(t-i)$ $\therefore \pm i$ are the complex eigenvalues

So one can solve

$(iI - A)|x\rangle = |0\rangle$ has a solution $|x\rangle = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$(-iI - A)|y\rangle = |0\rangle$ has a solution $|y\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\left[iI - A \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right] = \left[\begin{matrix} i & -1 \\ 1 & i \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right]$

$\left[-i - A \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right] = \left[\begin{matrix} -i & -1 \\ 1 & -i \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right] \to \left[\begin{matrix} -i & 1 \\ 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right]$

Proof $A \in M_n$. Want to show there is U unitary such that $U^\dagger A U = \begin{bmatrix} T_{11} & * \\ 0 & T_{nn} \end{bmatrix}$.

By induction on n.
When $n=1$, clear. $A = [A_{11}] = [1][A_{11}][1]$ is upper triangular form.
Assume the result holds for $B \in M_{n-1}$, $n-1 \geq 1$.
Consider A and $\det(tI-A) = 0$ so that we get $A|\lambda\rangle = \lambda|\lambda\rangle$.
We may replace $|\lambda\rangle$ by $\frac{|\lambda\rangle}{\||\lambda\rangle\|}$ so that $|\lambda\rangle$ has length 1.
Then I can find U_1 unitary with first column equal to $|\lambda\rangle$.
Now. $AU_1 = [\lambda|\lambda\rangle \mid A|x_2\rangle \mid \cdots A|x_n\rangle]$ ~~$U_1^\dagger A U_1 = [\lambda$~~

$$U_1^\dagger A U_1 = \begin{bmatrix} \langle\lambda| \\ \langle x_2| \\ \vdots \\ \langle x_n| \end{bmatrix} \Big[A|\lambda\rangle \,\Big|\, A|x_2\rangle \,\Big|\, \cdots \, A|x_n\rangle \Big]$$

$$= \begin{bmatrix} \lambda & * & \cdots & * \\ 0 & & & \\ 0 & & A_1 & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \quad A_1 \in M_{n-1}$$

$\lambda\langle\lambda|\lambda\rangle$

$\lambda\langle x_2|\lambda\rangle = 0$

$\lambda\langle x_3|\lambda\rangle = 0$

Now, by induction assumption, there is unitary $U_2 \in M_{n-1}$ s.t.,

$$U_2^\dagger A_1 U_2 = \begin{bmatrix} T_{22} & & * \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix} \in M_{n-1}.$$

Let $\quad U = U_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2 \end{array}\right] \in M_n.$

Then

$$U^\dagger A U = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2^\dagger \end{array}\right] U_1^\dagger A U_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2 \end{array}\right]$$

$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2^\dagger \end{array}\right] \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & A_1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & U_2^\dagger A_1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda & ** \\ \hline 0 & U_2^\dagger A_1 U_2 \end{array}\right] = \left[\begin{array}{c|ccc} \lambda & & ** & \\ \hline 0 & T_{22} & & * \\ \vdots & & \ddots & \\ 0 & 0 & & T_{nn} \end{array}\right].$$

$\blacksquare$

Corollary. $U^\dagger A U = [T_{ij}]$ is normal

Triangular form

if and only if $[T_{ij}] = \begin{bmatrix} T_{11} & & 0 \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix}$

diagonal matrix

Proof: $(\Leftarrow)$

If $A = U[T_{ij}]U^\dagger = U \begin{bmatrix} T_{11} & & 0 \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix} U^\dagger$

then $AA^\dagger = U \begin{bmatrix} T_{11} & & 0 \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix} U^\dagger \underbrace{\left(U \begin{bmatrix} T_{11} & & 0 \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix} U^\dagger \right)}_{A}{}^\dagger$

$$= U \begin{bmatrix} T_{11} & & 0 \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix} U^\dagger \left(U \begin{bmatrix} T_{11} & & 0 \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix}^\dagger U^\dagger \right)$$

$$= U \begin{bmatrix} |T_{11}|^2 & & 0 \\ & \ddots & \\ 0 & & |T_{nn}|^2 \end{bmatrix} U^\dagger$$

Similarly

$$A^\dagger A = U \begin{bmatrix} |T_{11}|^2 & & 0 \\ & \ddots & \\ 0 & & |T_{nn}|^2 \end{bmatrix} U^\dagger = AA^\dagger.$$