§1.10 Tensor product (Kronecker product) $m \times n$ $\quad p \times q$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad m \times n$ $\quad B$

$A = (A_{ij})$ $\quad B$

$A \otimes B : mp \times nq$

$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \}2m$

$2n$

- $A \otimes B = (A_{ij}B)$ satisfies
  (a) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,
  (b) $A \otimes (B + C) = A \otimes B + A \otimes C$, $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
  (c) $\mathrm{tr}\,(A \otimes B) = (\mathrm{tr}\,A)(\mathrm{tr}\,B)$ and $\det(A \otimes B) = \det(A)^n \det(B)^m$ if $A \in M_m, B \in M_n$.

Proof. (a) Let A be $m \times n$, B be $r \times s$, C be $n \times p$, D be $s \times q$. Then $A \otimes B = (A_{ij}B)$ is $mr \times ns$, and $C \otimes D = (C_{ij}D)$ is $ns \times pq$. Now

$$(A \otimes B)(C \otimes D) = (A_{ij}B)(C_{ij}D) = (f_{rs}BD),$$

$\begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \begin{bmatrix} C_{11}D & C_{12}D & C_{13}D \\ C_{21}D & C_{22}D & C_{23}D \end{bmatrix}$

$= [(a_{11}C_{11} + a_{12}C_{21})BD \quad BD$

where $f_{rs} = \sum_{\ell=1}^{n} A_{r\ell}C_{\ell s} = (AC)_{rs}$. So, $(f_{rs}BD) = AC \otimes BD$.

(b) $A \otimes (B + C) = (A_{ij}(B + C)) = (A_{ij}B) + (A_{ij}C) = A \otimes B + A \otimes C$.

$(A \otimes B)^\dagger = \overline{A \otimes B)^t = (A^t \otimes B^t)} = A^\dagger \otimes B^\dagger$.

$(A^{-1} \otimes B^{-1})(A \otimes B) = (A^{-1}A) \otimes (B^{-1}B) = I_m \otimes I_n = I_{mn} \implies A^{-1} \otimes B^{-1} = (A \otimes B)^{-1}$.

(c) Note that $\mathrm{tr}\,(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}B_{ji} = \mathrm{tr}\,(BA)$. Assume that A is $m \times m$ and B is $n \times n$. Let U, V be unitary and S, T be triangular such that $A = USU^\dagger$ and $B = VTV^\dagger$. Then $\mathrm{tr}\,A = \mathrm{tr}\,S, \mathrm{tr}\,B = \mathrm{tr}\,T$,

$$\mathrm{tr}\,(A \otimes B) = \mathrm{tr}\,((U \otimes V)(S \otimes T)(U \otimes V)^\dagger) = \mathrm{tr}\,(S \otimes T)$$

$$= \sum_{i=1}^{m} S_{ii} \sum_{j=1}^{n} T_{jj} (\sum_{i=1}^{m} S_{ii})(\sum_{j=1}^{n} T_{jj}) = (\mathrm{tr}\,S)(\mathrm{tr}\,T) = (\mathrm{tr}\,A)(\mathrm{tr}\,B).$$

Also,

$$\det(A \otimes B) = \det((U \otimes V)(S \otimes T)(U \otimes V)^\dagger) = \det(S \otimes T)$$

$$= \prod_{i=1}^{m} {S_{ii}}^n \det(T) = (\det S)^n (\det T)^m = (\det A)^n (\det B)^m.$$

$$A \otimes B = (A_{ij} B)$$

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad B^{m \times n}$$

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & a_{13} B \\ a_{21} B & a_{22} B & a_{23} B \end{pmatrix} \Big\} 2m$$

$$3n$$

4×6

(a) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

$C = 3 \times p \qquad D = 2 \times q.$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \qquad B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$(A \otimes B)(C \otimes D)$$

$$= \begin{bmatrix} A_{11} B & A_{12} B & a_{13} B \\ A_{21} B & a_{22} B & a_{23} B \end{bmatrix} \begin{bmatrix} C_{11} D & C_{12} D & \cdots & C_{1p} D \\ C_{21} D & C_{22} D & \cdots & C_{2p} D \\ C_{31} D & C_{32} D & \cdots & C_{3p} D \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} C_{11} BD + A_{12} C_{21} BD + A_{13} C_{31} BD & \cdots & A_{11} C_{1p} BD + \cdots + A_{13} C_{3p} BD \\ A_{21} C_{11} BD + A_{22} C_{21} BD + \cdots & \cdots & \end{bmatrix}$$

$$= \begin{bmatrix} f_{11} BD & f_{12} BD & \cdots & f_{1p} BD \\ f_{21} BD & f_{22} BD & & f_{2p} BD \end{bmatrix}$$

$$= F \otimes BD$$

$$= (AC) \otimes (BD)$$

(b.1) $$A\otimes(B+C) = \left(A_{ij}(B+C)\right)$$
$$= \left(A_{ij}B\right) + \left(A_{ij}C\right)$$
$$= A\otimes B + A\otimes C$$

(b.2) $$(A\otimes B)^{\dagger} = \left(A_{ij}B\right)^{\dagger}$$
$$= \left(\overline{A_{ij}B}\right)^{t}$$
$$= \left(\overline{A_{ij}}\;\overline{B}\right)^{t}$$

$$= \left(\cancel{A_{ij}C_{ij}}^{\,t}\;\overline{C_{ij}}\,B\right)$$

where $C_{ij} = A_{ji}$

$A^{t} = C$

$$= A^{\dagger}\otimes B^{\dagger}$$

$C_{ij} = A_{ji}$

(b.3) $$\left(\overset{m\times m}{A}\otimes\overset{n\times n}{B}\right)^{-1} = A^{-1}\otimes B^{-1}$$

Check $$(A\otimes B)(A^{-1}\otimes B^{-1})$$
$$= (A\cdot A^{-1})\otimes(B\cdot B^{-1}) = I_m\otimes I_n = I_{mn}$$

Rmk: ① $(A\otimes B)\cdot(C\otimes D)\cdot(E\otimes F) = (ACE)\otimes(BDF)$

② $A|u\rangle = a_i|u\rangle$ $\quad B|v\rangle = b_i|v\rangle$ $\quad (A\otimes B)(|u\rangle\otimes|v\rangle) = (A|u\rangle)\otimes(B|v\rangle)$

② $A|u\rangle = a_1|u\rangle \quad B|v\rangle = b_1|v\rangle$

$$(A\otimes B)(|u\rangle\otimes|v\rangle) = (A|u\rangle)\otimes(B|v\rangle)$$
$$= (a_1|u\rangle)\otimes(b_1|v\rangle)$$
$$= (a_1b_1)(|u\rangle\otimes|v\rangle)$$

$|uv\rangle$

---

③ $A = USU^\dagger \qquad S = \begin{bmatrix} \ddots & \\ 0 & \end{bmatrix}$

$B = VTV^\dagger \qquad T = \begin{bmatrix} \ddots & \\ 0 & \end{bmatrix}$

U, V unitary

$$\underline{\underline{A\otimes B}} = (USU^\dagger)\otimes(VTV^\dagger)$$
$$= (U\otimes V)(S\otimes T)(U\otimes V)^\dagger$$

$$S\otimes T = \begin{bmatrix} s_{11}T & s_{12}T & \cdots & \\ & s_{22}T & & \cdots \\ 0 & & \ddots & \\ & & & s_{mm}T \end{bmatrix}$$

$$tr(A\otimes B) = s_{11}trT + s_{22}trT + \cdots + s_{mm}trT = (trS)(trT)$$

Note $\quad A$ $m\times m$ $\quad \underline{tr(A) = \sum_{i=1}^{m} A_{ii}}, \quad tr(XY) = tr(YX)$

$$\text{"}\sum_{i,j} X_{ij}Y_{ji}\text{"}$$

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ X & & \end{pmatrix}$$

$$trA = tr\, USU^\dagger.$$
$$= tr\, SU^\dagger U$$
$$= trS.$$

$$\det(A\otimes B) = \det(S\otimes T)$$
$$= \det(s_{11}T)\cdots\det(s_{mm}T)$$
$$= s_{11}^n\det(T) \cdots s_{mm}^n\det(T) = (s_{11}\cdots s_{mm})^n \det(T)^m$$

$$\det(A)^n\det(B)^m = \det(S)^n\det(T)^m = (s_{11}\cdots s_{mm})^n\det(T)^m$$

Chapter 2 Quantum Mechanics

Quantum Information Science uses quantum properties to help store, process, and transmit information. In this chapter, we describe some basic background on quantum mechanics. We first use vector states to describe quantum systems. Then we demonstrate the formulation using density matrices.

$0, 1 \in \mathbb{Z}_2$ $\quad |0\rangle, |1\rangle \in \mathbb{C}^2$

Copenhagen interpretation

A1 A vector state $|x\rangle$ is a $\boxed{\text{unit vector}}$ in a Hilbert space $\mathcal{H}$ (usually $\mathbb{C}^n$). Linear combinations (superposition) of the physical states are allowed in the state space.

A2 Every physical quantity (observable) corresponds to a Hermitian operator (matrix) A. Suppose a state $|x\rangle = c_1|u_1\rangle + c_2|u_2\rangle$ such that $A|u_i\rangle = a_i|u_i\rangle$ for $i \in \{1, 2\}$. Then applying a measurement of $|x\rangle$ corresponding to A will cause the **wave function** (that describes the quantum state) to **collapse** to $|u_1\rangle$ or $|u_2\rangle$ with probability of $|c_1|^2$ and $|c_2|^2$, respectively. Here c_1, c_2 are called the probability amplitude of the state $|x\rangle$.

$\sigma_z = A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

A3 The time dependence of a state is governed by the Schrödinger equation

$$i\hbar \frac{\partial |x\rangle}{\partial t} = H|x\rangle,$$

$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

where $\hbar$ is the Planck constant with

$$\hbar = 6.6260700410^{-34} m^2 kg/s,$$

and H is a Hermitian operator (matrix) corresponding to the energy of the system known as the Hamiltonian.

A1 $\quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A2 $\quad A|x\rangle$: $\quad a_1 |u_1\rangle \longrightarrow |u_1\rangle$, $\quad a_2 |u_2\rangle \longrightarrow |u_2\rangle$

A3 $\quad i\hbar \dfrac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = H(t) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$\begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix}$

If $H = H^+$ is a constant,

$|x(t)\rangle = e^{iHt} |x(0)\rangle$

$|x\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 |0\rangle + x_2 |1\rangle$

$|x_1|^2 + |x_2|^2 = 1.$

$\dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} \quad \dfrac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\sqrt{2}}$

$U = \left[\dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} \dfrac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\sqrt{2}} \right]$

$A = U \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U^+$

$|x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ $\quad |\frac{1}{\sqrt{2}}|^2$ prob $\begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\frac{1}{\sqrt{2}}|^2$ prob $\begin{pmatrix}0\\1\end{pmatrix}$

But $|x\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle \neq \alpha_1|u_1\rangle + \alpha_2 e^{it}|u_1\rangle$

Remarks

1. The phase of the state does not matter, i.e., $|x\rangle$ and $e^{i\alpha}|x\rangle$ represents the same states.
2. In the finite dimensional case, if the state and the observable are represented by

$$|x\rangle = \sum_{j=1}^{n} c_j|u_j\rangle \in \mathbb{C}^n \quad \text{and} \quad A = \sum_{j=1}^{n} \lambda_j|u_j\rangle\langle u_j| = \sum_{j=1}^{n} \lambda_j P_j,$$

$= \sum_{\alpha} \alpha P_\alpha$

then the projective measurement of the state is

$$\langle x|A|x\rangle = \boxed{\sum_{j=1}^{n} \lambda_j|c_j|^2.}$$

Once the measurement is applied, the state becomes (collapses to)

$$\frac{P_i|x\rangle}{\sqrt{\langle x|P_i|x\rangle}} = \frac{P_i|x\rangle}{|c_i|}.$$

$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \to \frac{x_1\begin{pmatrix}1\\0\end{pmatrix}}{|x_1|}$, $\to x_2\begin{pmatrix}0\\1\end{pmatrix}$

3. In the Schrödinger equation, if $H(t)$ does not depend on t, then

$$|x(t)\rangle = e^{-iHt/\hbar}|x(0)\rangle.$$

Otherwise,

$$\boxed{|x(t)\rangle = \exp\left(\frac{-i}{\hbar}\int_0^t H(s)ds\right)|x(0)\rangle.}$$

Example If

$$H = \frac{-\hbar}{2}w\sigma_x \quad \text{and} \quad |\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{so that} \quad i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle,$$

then $|\psi(t)\rangle = \exp(i\frac{w}{2}\sigma_x t)|\psi(0)\rangle$. Hence,

$$|\psi(t)\rangle = ((\cos wt/2)I_2 + (i\sin wt/2))\sigma_x|\psi(0)\rangle = \begin{pmatrix}\cos wt/2\\ i\sin wt/2\end{pmatrix}.$$