Continued Fraction decomposition of rational numbers

$$\frac{13}{57} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 +}}}$$

$$= 0 + \cfrac{1}{\frac{57}{13}}$$

$$= 0 + \cfrac{1}{4 + \frac{5}{13}}$$

$$\begin{array}{r} 4 \\ 13\overline{)57} \\ \underline{52} \\ 5 \end{array}$$

$$= 0 + \cfrac{1}{4 + \cfrac{1}{\frac{13}{5}}}$$

$$= 0 + \cfrac{1}{4 + \cfrac{1}{2 + \frac{3}{5}}}$$

$$= 0 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{3}}}}$$

$$= 0 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{2}{3}}}}$$

$$= 0 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}}}}$$

$$= 0 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}}$$

[0, 4, 2, 1, 1, 2]

Example 8.2 p.152

⓪ $Q = 2^{20} = 1048576$

$\boxed{\frac{1}{2Q} = \frac{1}{2\cdot 2^{20}} \triangleq 4.76837\times 10^{-7}}$

$\frac{y}{p}$ $y = \frac{c}{p}$

① $[0, 122, 1, 2, 44, 5, 3]$

$$\frac{799}{1048576} = 0 + \cfrac{1}{122 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{44 + \cfrac{1}{5 + \cfrac{1}{3}}}}}}$$

$[a_0 \cdots]$

$= [0, 122, 1, 3, 44, 5, 3]$

② $p_0 = a_0 = 0, \quad q_0 = 1.$

③ $p_1 = a_1 p_0 + 1 = 122 \times 0 + 1 = 1$

$q_1 = a_1 q_0 = 122 \times 1 = 122$

y/Q

$\left|\frac{p_1}{q_1} - \frac{8548}{1048576}\right| = 2.19\times 10^{-5} > \frac{1}{2Q}$

④ $p_2 = a_2 p_1 + p_0 = 1, \quad q_2 = a_2 q_1 + q_0 = 123$

$\left|\frac{p_2}{q_2} - \frac{y}{Q}\right| \quad \frac{p_2}{q_2} = \left|\frac{1}{123} - \frac{y}{Q}\right| = 2.19\times 10^{-5} > \frac{1}{2Q}$

⑤ $p_3 = a_3 p_2 + p_1 = 3, \quad q_3 = a_3 q_2 + q_1 = 368$

$\left|\frac{p_3}{q_3} - \frac{y}{Q}\right| = \left|\frac{3}{368} - \frac{y}{Q}\right| = \cancel{2.19}\ 1.65856 \times 10^{-7} < \frac{1}{2Q}$

⑥ $P = q_3 = 368$

$\frac{A|\psi\rangle\langle\psi|A^{+}}{tr(A|\psi\rangle\langle\psi|A^{+})}$

9.1 Open Quantum System

A unitary time evolution of a close system is determined by the quantum map $\mathcal{E}$ defined by

$$\boxed{\mathcal{E}(\rho_S) = U(t)\rho_S U(t)^\dagger.}$$

$|\psi_t\rangle = U_t|\psi_0\rangle$ $|\psi\rangle = \alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$

Here, ρ_S is the density matrix of a closed system at time $t = 0$ and $U(t)$ is the time evolution operator.

$\rho = t|\psi_1\rangle\langle\psi_1| + (1-t)|\psi_2\rangle\langle\psi_2|$

An open system is a system of interest (called the **principal system**) coupled with its environment. The total Hamiltonian is given by

$$H_T = H_S + H_E + E_{SE},$$

where H_S, H_E and H_{SE} are the system Hamiltonian, the environment Hamiltonian and their interaction Hamiltonian, respectively.

The state of the total system, which is assumed to be closed, will be described by ρ acting on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$ such that

$$\boxed{\rho(0) = \rho_S \otimes \rho_E \quad \text{and} \quad \rho(t) = U(t)(\rho_S \otimes \rho_E)U(t)^\dagger \text{ for } t > 0.}$$

$\in M_{mn}$ $= (\#)^n$

We study the system ($\mathcal{H}_S$) by taking the partial trace

$$\rho_S(t) = \mathrm{Tr}_E[U(t)(\rho_S \otimes \rho_E)U(t)^\dagger].$$

$\in M_m$ $(I \otimes \langle\varepsilon_i|)$ Example: $\rho_S \in M_m$ $(I_m \otimes \langle\varepsilon_i|)$ $n \times n$ mn $mn \times mn$ $mn \times m$

We may assume $\rho_E = |\varepsilon_0\rangle\langle\varepsilon_0|$ by purification. Then

$$\rho_S(t) = \sum_a E_a(t)\rho_S E_a(t)^\dagger \quad \text{with} \quad E_a(t) = \langle\varepsilon_a|U(t)|\varepsilon_0\rangle. = \langle I \otimes \varepsilon_a| U(t) | I \otimes \varepsilon_0\rangle$$

M_m m $m\,n$

This is known as the operator-sum representation of the quantum operation $\mathcal{E}$. The operators $E_a(t)$ are known as the **Kraus operators** of the quantum operation $\mathcal{E}$. Note that

$$\sum_a E_a(t)^\dagger E_a(t) = I.$$

It is also possible to define a quantum operation such that $\rho_E \to \rho_S(t)$ such as

$$\rho_S(t) = \mathrm{Tr}_E[U(t)(\varepsilon_0)\langle\varepsilon_0| \otimes \rho_E)U(t)^\dagger].$$

$$\mathrm{Tr}_E(U(t)(\rho_S \otimes \rho_E)U(t)^\dagger) = \sum_{i=1}^{n}(I_m \otimes \langle e_i|)(\qquad)|I_m \otimes e_i\rangle$$

$|\varepsilon_0\rangle\langle\varepsilon_0|$

$\boxed{\text{In practice}}$, for any quantum operation $\Phi: M_m \to M_n$ we can "find" $E_r(t) \in M_{n,m}$ s.t. $\Phi(\rho_S) = \sum_{a=1} E_a(t)\rho_S E_a(t)^\dagger \in M_n$ $\sum E_a(t)^\dagger E_a(t) = I_m$

$\sigma_x|0\rangle = |1\rangle$

$\sigma_x|1\rangle = |0\rangle$

$\begin{bmatrix}1&0\\0&0\end{bmatrix} = |0\rangle\langle 0| \longrightarrow \begin{bmatrix}0&0\\0&1\end{bmatrix} = |1\rangle\langle 1|$

$\begin{bmatrix}0&0\\0&1\end{bmatrix} \mapsto \begin{bmatrix}1&0\\0&0\end{bmatrix} = |0\rangle\langle 0|$

9.3 Examples of Quantum Channels

Bit-Flip Channel

Define the bit-flip channel by

$\begin{bmatrix}0&1\\1&0\end{bmatrix}\rho\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$$\boxed{\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_x\rho_S\sigma_x, \quad 0 \le p \le 1.}$$

The input state ρ_S is bit-flipped $(|0\rangle, |1\rangle) \mapsto (|1\rangle, |0\rangle$ with a probability p, and remains unchanged with a probability $1-p$. The Choi/Kraus operators are: $E_0 = \sqrt{1-p}I$ and $E_1 = \sqrt{p}\sigma_x$.

This can be modeled by

$$\rho_S \mapsto V(\rho_S \otimes [(1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|])V^t = (1-p)\rho_S \otimes |0\rangle\langle 0| + p\,\sigma_x\rho_S\sigma_x|1\rangle\langle 1|$$

with $V = I \otimes |0\rangle\langle 0| + \sigma_x|1\rangle\langle 1|$ so that

$\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_x\rho_S\sigma_x.$$

We may also use $\rho_E = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle$.

$\rho_S = \frac{1}{2}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}c_z&0\\0&-c_z\end{bmatrix} + \begin{bmatrix}0&c_x\\c_x&0\end{bmatrix} + \begin{bmatrix}0&-ic_y\\ic_y&0\end{bmatrix}\right)$

Use the block sphere representation:

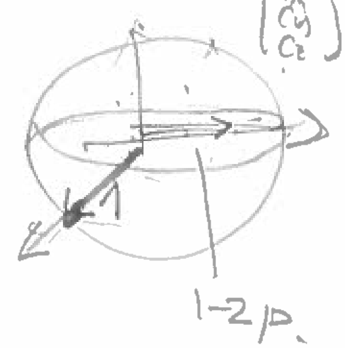
$$\rho_S = \frac{1}{2}\left(I + \sum_{k=x,y,z} c_k\sigma_k\right), \quad c_x^2 + c_y^2 + c_z^2 \le 1.$$

Then

$$\mathcal{E}(\rho_S) = \frac{1}{2}\left(I + c_x\sigma_x + (1-2p)c_y\sigma_y + (1-2p)c_z\sigma_z\right).$$

So, the block sphere shrinks in the y and z directions by a factor of $|1-2p|$.

$\begin{bmatrix}c_x\\c_y\\c_z\end{bmatrix}$



1-2p.

$\frac{\begin{pmatrix}1&1\\1&1\end{pmatrix}}{2} = (1-p)\frac{\begin{bmatrix}1&1\\1&1\end{bmatrix}}{2} + p\sigma_x\frac{\begin{bmatrix}1&1\\1&1\end{bmatrix}}{2}\sigma_x = \mathcal{E}\left(\frac{1}{2}(I+\sigma_x)\right)$

$\begin{pmatrix}1&(1-p)(-i)+pi\\ &1\end{pmatrix} = (1-p)\frac{\begin{bmatrix}1&-i\\i&1\end{bmatrix}}{2} + p\sigma_x\frac{\begin{bmatrix}1&-i\\i&1\end{bmatrix}}{2}\sigma_x = \mathcal{E}\left(\frac{1}{2}(I+\sigma_y)\right)$

$\mathcal{E}\left(\frac{1}{2}(I+\sigma_z)\right)$

$(1-p)i + p(-i)$

$= \begin{pmatrix}1&(1-2p)(-i)\\(1-2p)i&1\end{pmatrix}$

Examine $\mathcal{E}\left(\frac{1}{2}I + \begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)$