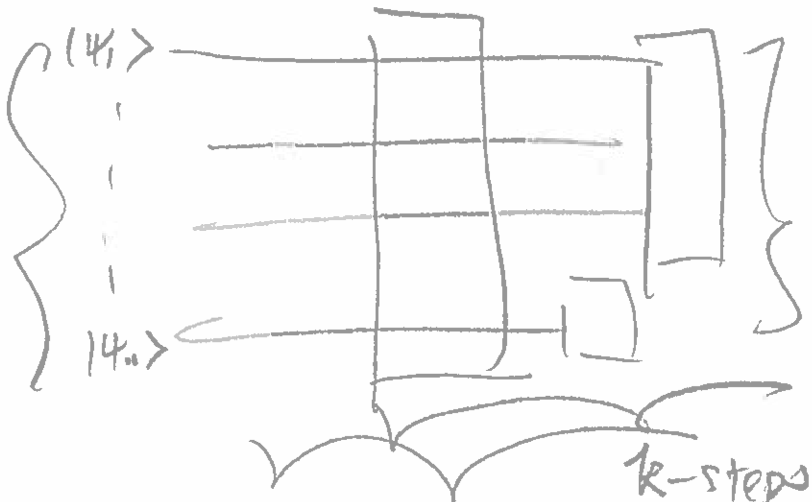
Quantum Circuit: & the corresponding quantum Operation



$$\underbrace{U_k \cdots U_1}_{2^n} (\underbrace{|\psi_1\rangle \cdots |\psi_n\rangle}) = |\varphi_1 \cdots \varphi_n\rangle$$

Corresponding quantum Operation:

$$\mathcal{E}(\underline{|\psi\rangle\langle\psi|}) = |\varphi\rangle\langle\varphi|$$

defined by

$$\mathcal{E}(|\psi\rangle\langle\psi|) = U|\psi\rangle\langle\psi| U^{+}$$

where $U = U_k \cdots U_1$

Extend to mixed states

$$\mathcal{E}(\rho) = U \rho U^{+}$$

Bit-Flip

Circuit diagram

$$|\psi\rangle \;—\; \oplus \;—\; |\psi_0\rangle$$

Quantum channel.

$$\mathcal{E}(\rho) = \sigma_x \rho \sigma_x^\dagger$$

$$U_t(\rho_S \otimes \rho_E)\,U_t^\dagger = \rho_S(t) \otimes \rho_E(t)$$

$$V_1(t) \otimes V_2(t)\,(\quad)\,(V_1(t) \otimes V_2(t))^\dagger$$

$$\| $$

$$\left(V_1(t)\,\rho_S\,V_1(t)^\dagger\right) \otimes \left(V_2(t)\,\rho_E\,V_2(t)^\dagger\right)$$

Noisy quantum channels are quantum operations.

For example, suppose U_a is unitary, $p_a \in (0,1]$, and $\sum_a p_a = 1$. A **mixing process** is defined by

$$\boxed{\mathcal{M}(\rho_S) = \sum_a p_a U_a \rho_S U_a^\dagger.} \longrightarrow \rho_S$$

Remark $\boxed{\text{The description of quantum operations and noisy quantum channels are interchangeable.}}$

Completely positive linear maps.

Definition A linear map (function) Λ on matrices such that $\Lambda \otimes I_r$ maps positive operators to positive operators is call **completely positive.**

Theorem A map on matrices is completely positive if and only if it admits an operator-sum representation:

$$A \mapsto \sum_j E_j A E_j^\dagger.$$

at a fixed time t

$\sum E_j^\dagger E_j = I$

The matrices E_j are called the Choi/Kraus operators. In the context of quantum error correction, E_j are called the error operators.

Remark: A linear map $\Phi : M_n \to M_m$ is positive if

$A \in M_n$ is positive semi-definite $\Rightarrow$ $\Phi(A)$ is positive semi-definite

Φ is 2-positive if $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{2n}$ is positive semidefinite

$$\Rightarrow \begin{pmatrix} \Phi(A_{11}) & \Phi(A_{12}) \\ \Phi(A_{21}) & \Phi(A_{22}) \end{pmatrix} \text{ is p.s.d}$$

Φ is k-positive if

$$nk \left\{ \overbrace{\begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}}^{nk} \right. \in M_k(M_n) \cong M_k \otimes M_n$$

is positive semi-definite

$$\Rightarrow \begin{pmatrix} \Phi(A_{11}) & \cdots & \Phi(A_{1k}) \\ \vdots & & \vdots \\ \Phi(A_{k1}) & \cdots & \Phi(A_{kk}) \end{pmatrix} \text{ is p.s.d}$$

Φ is Completely positive if it is k-positive $\forall k = 1, 2, 3, \ldots$

$A = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $P_1 = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ $P_2 = \begin{bmatrix}0&0\\0&1\end{bmatrix}$

$|\lambda_1\rangle\langle\lambda_1|$ $|\lambda_2\rangle\langle\lambda_2|$

9.2 Measurements as quantum operations

$$\frac{P_1\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}}{\left\|P_1\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}\right\|} = \frac{\begin{bmatrix}\alpha_1\\0\end{bmatrix}}{|\alpha_1|} = \begin{bmatrix}e^{i\theta_1}\\0\end{bmatrix}$$

$$\frac{P_2\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}}{\left\|P_2\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}\right\|} = \frac{\begin{bmatrix}0\\\alpha_2\end{bmatrix}}{|\alpha_2|} = \begin{bmatrix}0\\e^{i\theta_2}\end{bmatrix}$$

Projective measurements as quantum operations.

Let $A = \sum_j \lambda_j P_j$ so that the measurement of A in a state ρ is

$$p(j) = \mathrm{Tr}(P_j \rho P_j) = \mathrm{Tr}(P_j \rho)$$

$|\psi\rangle\langle\psi|$ $\mathrm{Tr}(P_j^2 \rho)$

and state is changed to

$$\rho \to P_j \rho P_j / p(j).$$

Note: $P_j = P_j^2 = P_j^\dagger$

Then the measurement process is the quantum operation

$$\rho_S \mapsto \sum_j p(j) \frac{P_j \rho_S P_j}{p(j)} = \sum_j P_j \rho_S P_j.$$

Positive Operator-valued measure (POVM)

Suppose $|\psi\rangle|e_0\rangle$ is the state of an open system. Let U be a unitary operator acting on the system such that

$$|\Psi\rangle = U|\psi\rangle|e_0\rangle = \sum_j M_j|\psi\rangle|e_j\rangle.$$

Consider $\{M_1, \ldots, M_r\}$

Then

$$1 = \langle e_0|\langle\psi|U^\dagger U|\psi\rangle|e_0\rangle = \langle\psi|\sum_j M_j^\dagger M_j|\psi\rangle = \sum_j M_j|\psi\rangle\langle\psi|M_j^\dagger.$$

Since $|\psi\rangle$ is arbitrary, we have $\sum_j M_j^\dagger M_j = I.$ In general, suppose M_j acts on $\mathcal{H}_S$ such that $\sum_j M_j^\dagger M_j = I$. Then $\{M_j^\dagger M_j\}$ forms a POVM and

$$\rho_S \mapsto \sum_j M_j \rho_S M_j^\dagger$$

is a quantum operation.

Example: $\mathcal{E}: M_2 \to M_2$

$\mathcal{E}(\rho) = (1-p)\rho + p\sigma_x\rho\sigma_x.$

Define $M_1 = \sqrt{1-p}\, I$

$M_2 = \sqrt{p}\,\sigma_x$

$M_1^\dagger M_1 + M_2^\dagger M_2 = I_2$

$\sigma_z \rho_s \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rho_s \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}$

Phase-Flip Channel

The phase-flip channel is defined by

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_z \rho_S \sigma_z, \qquad 0 \le p \le 1.$$

$\sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \sigma_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

The input state ρ_S is phased-flipped $(|0\rangle, |1\rangle) \mapsto (|0\rangle, -|1\rangle$ with a probability p, remains unchanged with a probability $1-p$. The Choi/Kraus operators are: $E_0 = \sqrt{1-p}I$ and $E_1 = \sqrt{p}\sigma_z$.

This can be modeled by

$$\rho_S \mapsto V(\rho_S \otimes [(1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|])V^\dagger = (1-p)\rho_S \otimes |0\rangle\langle 0| + p\,\sigma_z \rho_S \sigma_z |1\rangle\langle 1|$$

with $V = I \otimes |0\rangle\langle 0| + \sigma_z |1\rangle\langle 1|$ so that taking partial trace (to remove the environment effect) will give

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_x \rho_S \sigma_x.$$

We may also use $\rho_E = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle$.

Use the block sphere representation:

$$\mathcal{E}(\rho_S) = \frac{1}{2}\left(I + (1-2p)c_x\sigma_x + (1-2p)c_y\sigma_y + c_z\sigma_z\right).$$

So, the block sphere shrinks in the y and z directions by a factor of $|1-2p|$.