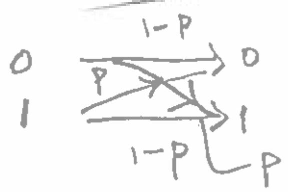
Quantum error correction:

$$|\psi\rangle = a|0\rangle + b|1\rangle \xrightarrow{\mathcal{E}} |\hat{\psi}\rangle \xrightarrow{\mathcal{C}} |\psi\rangle$$

Classical Classical:



Codewords

$0 \to (000)$

$1 \to (111)$

Repetition code

$$x \to (xxx) \xrightarrow{\mathcal{E}} (x_1 x_2 x_3) \xrightarrow{\mathcal{C}}$$

use maximum likelihood decoding

Quantum

$$|\psi\rangle = a|0\rangle + b|1\rangle \sim \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$$

$$|\psi\rangle_L \to a|000\rangle + b|111\rangle \neq |\psi\rangle|\psi\rangle|\psi\rangle$$

Remark:

$$|\psi\rangle|0\rangle|0\rangle \xrightarrow{\text{unitary}} a|000\rangle + b|111\rangle.$$

$$|\psi\rangle|0\rangle|0\rangle \not\to |\psi\rangle|\psi\rangle|\psi\rangle$$

Impossible by no-cloning.

We can now do the encoding, transmission, measurement, decoding as discussed before.

$$e^{i\alpha X} = I + \frac{i\alpha X}{1!} - \frac{\alpha^2 I}{2!} + \frac{-i\alpha^3 X}{3!} + \cdots$$

Continuous rotation

Suppose the error operator is

$$U_\alpha = e^{i\alpha X} = \cos\alpha I + i\sin\alpha X$$

~~error operat~~

and U_α acts on each qubit with a probability $p \in (0, 1/2)$.

Suppose we use the same QECC scheme for the bit-flip channel.

If U_α acts on the first logical qubit $\boxed{|\psi\rangle_L = a|000\rangle + b|111\rangle}$ then the transmitted state becomes

$$(U_\alpha \otimes I \otimes I)|\psi\rangle_L = \boxed{\cos\alpha|\psi\rangle_L + i\sin\alpha(a|100\rangle + b|011\rangle))}$$

Applying syndrome measurement $|AB\rangle$ to the transmitted state, we get

$$\cos\alpha|\psi\rangle|00\rangle + i\sin\alpha(a|100\rangle + b|011\rangle))|11\rangle,$$

Case 1. If we get $|00\rangle$, the first register collapses to $|\psi\rangle_L$, and no correction is needed.

Case 2. If we get $|11\rangle$, the first register collapses to $a|100\rangle + b|011\rangle$ and we may apply correction to recover $|\psi\rangle_L$.

Remark: If the error operator has the form

$$\mu_1 I + \mu_2 X.$$

then our error correction scheme works for the Bit-Flip error

$Y = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} = i\sigma_y$

$\sigma_x, \sigma_y, \sigma_z$

$\| \quad \| \quad \|$

$X \quad \quad Z$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$$

10.2 Phase-Flip QECC

One may consider the phase flip channel $\boxed{|x\rangle \mapsto Z|x\rangle = (-1)^x|x\rangle}$ for $x \in \{0, 1\}$.

$x = 0 \quad Z|0\rangle = |0\rangle$

$x = 1 \quad Z|1\rangle = -|1\rangle$

One may use the fact that $\boxed{U_H Z U_H} = \boxed{X}$ and adapt the QECC scheme of the bit-flip channel to the phase-flip channel.

One can also use the phase-flip QECC scheme to handle the continuous phase-flip channel

$$\boxed{|x\rangle \mapsto U_\beta|x\rangle = e^{i\beta Z}|x\rangle} \qquad \text{for } x \in \{0, 1\}.$$

$e^{i\beta Z} = (\cos\beta) I + (i\sin\beta) Z$

The probability of error becomes $P(error) = p\sin^2\alpha$.

Similar analysis can be done if U_α acts on other qubits.

$\mathcal{U} = \mu_1 I + \mu_2 Z$

Observe that $Y = \underline{ZX} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

If I can correct X, Z, then I can use it to correct Y, and therefore correct error operator of the form $\mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 Z$

10.3 Shor's Nine-Qubit Code

Consider $X = \sigma_x, Z = \sigma_z, Y = i\sigma_y = ZX$. They will induce the Bit-Flip, Phase-Flip, and Phase-and-Bit-Flip error on a vector state $|\psi\rangle$.

Every unitary $U \in M_2$ is a linear combination of I, X, Y, Z. If there is a QECC for errors induced by X, Y, Z, then it can be used to correct general error.

Let $|+\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$.

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\-1\end{bmatrix}$

QECC scheme

1. Encode $|\psi\rangle = a|0\rangle + b|1\rangle$ by

$$|\psi\rangle = a|0\rangle + b|1\rangle \to a|000\rangle + b|111\rangle \to\to a|+++\rangle + b|---\rangle = |\psi\rangle_L.$$

2. Transmit the logical quit. The probability of one or no erros is $(1-p)^9 + 9p(1-p)^8 = (1+8p)(1-p)^8$ so that the probability of 2 or more error is

$$1 - (1+8p)(1-p)^2 = 36p^2 + O(p^3),$$

which is small if $p > 0$ is small.

3. Syndrome detection and correction.

Send in 6 ancillas in the first round to detect Bit-Flip error and apply correction.

Then send in 2 more ancillas to detect Phase-Flip error and apply correction.

Remark A QECC for error operators I, X, Y, Z corrects every single qubit error.