Quantum error correction without syndrome measurement

Chi-Kwong Li
Department of Mathematics
The College of William and Mary

Introduction

- M_n (H_n): the set of $n \times n$ complex (Hermitian) matrices.

Introduction

- M_n (H_n): the set of $n \times n$ complex (Hermitian) matrices.
- Quantum states are represented as density matrices, i.e., positive semidefinite matrices in H_n in with trace one.

Introduction

- M_n (H_n): the set of $n \times n$ complex (Hermitian) matrices.
- Quantum states are represented as density matrices,
  i.e., positive semidefinite matrices in H_n in with trace one.
- Denote by D_n the set of density matrices in M_n.

Introduction

- M_n (H_n): the set of $n \times n$ complex (Hermitian) matrices.
- Quantum states are represented as density matrices,
  i.e., positive semidefinite matrices in H_n in with trace one.
- Denote by D_n the set of density matrices in M_n.
- A quantum channel (operation) $\mathcal{E} : M_n \to M_n$ is a trace preserving completely positive map admitting the following operator sum representation

$$\mathcal{E}(\rho) = \sum_{j=1}^{r} F_j \rho F_j^{\dagger},$$

for some $F_1, \dots, F_r \in M_n$ such that $\sum_{j=1}^{r} F_j^* F_j = I_n$.

Introduction

- M_n (H_n): the set of $n \times n$ complex (Hermitian) matrices.
- Quantum states are represented as density matrices,
  i.e., positive semidefinite matrices in H_n in with trace one.
- Denote by D_n the set of density matrices in M_n.
- A quantum channel (operation) $\mathcal{E} : M_n \to M_n$ is a trace preserving completely positive map admitting the following operator sum representation

$$\mathcal{E}(\rho) = \sum_{j=1}^{r} F_j \rho F_j^{\dagger},$$

  for some $F_1, \ldots, F_r \in M_n$ such that $\sum_{j=1}^{r} F_j^* F_j = I_n$.
- In the context of quantum error correction, $F_1, \ldots, F_r$ are the error operators associated with the channel.

Quantum Error Correction

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with "known" error operators.

Quantum Error Correction

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with "known" error operators.

Some strategies

Quantum Error Correction

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with “known” error operators.

Some strategies

- Encode the data bits $\tilde{\rho} \in D_k$ as $\rho \in D_n$ for a larger n.

Quantum Error Correction

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with "known" error operators.

Some strategies

- Encode the data bits $\tilde{\rho} \in D_k$ as $\rho \in D_n$ for a larger n.
- The encoded state ρ goes through the channel and becomes $\mathcal{E}(\rho)$.

Quantum Error Correction

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with "known" error operators.

Some strategies

- Encode the data bits $\tilde{\rho} \in D_k$ as $\rho \in D_n$ for a larger n.
- The encoded state ρ goes through the channel and becomes $\mathcal{E}(\rho)$.
- Either apply a syndrome measurement using additional qubits;

Quantum Error Correction

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with "known" error operators.

Some strategies

- Encode the data bits $\tilde{\rho} \in D_k$ as $\rho \in D_n$ for a larger n.
- The encoded state ρ goes through the channel and becomes $\mathcal{E}(\rho)$.
- Either apply a syndrome measurement using additional qubits;
  then use a suitable quantum operation $\mathcal{R}$ to recover ρ, i.e., $\mathcal{R} \circ \mathcal{E}(\rho) = \rho$.

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with "known" error operators.

Some strategies

- Encode the data bits $\tilde{\rho} \in D_k$ as $\rho \in D_n$ for a larger n.
- The encoded state ρ goes through the channel and becomes $\mathcal{E}(\rho)$.
- Either apply a syndrome measurement using additional qubits;
  then use a suitable quantum operation $\mathcal{R}$ to recover ρ, i.e., $\mathcal{R} \circ \mathcal{E}(\rho) = \rho$.
- Or, use the Knill-Laflamme condition to construct a recovery channel $\mathcal{R}$ such that

  $$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \text{ whenever } \rho \in M_n \text{ has range space inside } V$$

  for a certain subspace V in $\mathbb{C}^n$.

Quantum Error Correction

Basic Problem

How to recover useful/important information (data bits) after it passes through a noisy quantum channel $\mathcal{E}$ with "known" error operators.

Some strategies

- Encode the data bits $\tilde{\rho} \in D_k$ as $\rho \in D_n$ for a larger n.
- The encoded state ρ goes through the channel and becomes $\mathcal{E}(\rho)$.
- Either apply a syndrome measurement using additional qubits;
  then use a suitable quantum operation $\mathcal{R}$ to recover ρ, i.e., $\mathcal{R} \circ \mathcal{E}(\rho) = \rho$.
- Or, use the Knill-Laflamme condition to construct a recovery channel $\mathcal{R}$ such that

  $$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \text{ whenever } \rho \in M_n \text{ has range space inside } V$$

  for a certain subspace V in $\mathbb{C}^n$.
- The subspace V is called a quantum error correction code (QECC) of the channel $\mathcal{E}$.

Example: Bit-flip channel

For the bit-flip error which changes $a|0\rangle + b|1\rangle$ to $a|1\rangle + b|0\rangle$ with a certain probability.

Example: Bit-flip channel

For the bit-flip error which changes $a|0\rangle + b|1\rangle$ to $a|1\rangle + b|0\rangle$ with a certain probability. We can use 2 ancillas to protect 1-qubit information by letting

$$V = \text{span}\{|000\rangle, |111\rangle\} \in (\mathbb{C}^2)^{\otimes 3}$$

be a QECC for the bit-flip channel $\mathcal{E} : M_8 \to M_8$:

Example: Bit-flip channel

For the bit-flip error which changes $a|0\rangle + b|1\rangle$ to $a|1\rangle + b|0\rangle$ with a certain probability. We can use 2 ancillas to protect 1-qubit information by letting

$$V = \text{span}\{|000\rangle, |111\rangle\} \in (\mathbb{C}^2)^{\otimes 3}$$

be a QECC for the bit-flip channel $\mathcal{E} : M_8 \to M_8$:

$$\mathcal{E}(\rho) = p_0\rho + p_1 X_1 \rho X_1^\dagger + p_2 X_2 \rho X_2^\dagger + p_3 X_3 \rho X_3^\dagger,$$

where $X_1 = \sigma_x \otimes I_2 \otimes I_2$, etc.

Example: Bit-flip channel

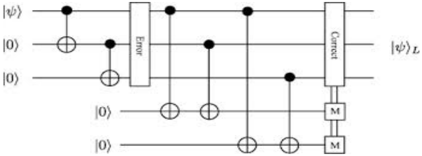
For the bit-flip error which changes $a|0\rangle + b|1\rangle$ to $a|1\rangle + b|0\rangle$ with a certain probability. We can use 2 ancillas to protect 1-qubit information by letting

$$V = \text{span}\{|000\rangle, |111\rangle\} \in (\mathbb{C}^2)^{\otimes 3}$$

be a QECC for the bit-flip channel $\mathcal{E} : M_8 \to M_8$:

$$\mathcal{E}(\rho) = p_0\rho + p_1 X_1 \rho X_1^\dagger + p_2 X_2 \rho X_2^\dagger + p_3 X_3 \rho X_3^\dagger,$$

where $X_1 = \sigma_x \otimes I_2 \otimes I_2$, etc.



QECC with syndrome measurement.

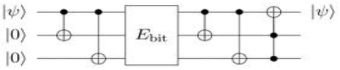
For the bit-flip error which changes $a|0\rangle + b|1\rangle$ to $a|1\rangle + b|0\rangle$ with a certain probability. We can use 2 ancillas to protect 1-qubit information by letting

$$V = \text{span}\{|000\rangle, |111\rangle\} \in (\mathbb{C}^2)^{\otimes 3}$$

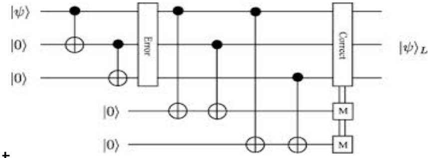
be a QECC for the bit-flip channel $\mathcal{E} : M_8 \to M_8$:

$$\mathcal{E}(\rho) = p_0\rho + p_1 X_1 \rho X_1^\dagger + p_2 X_2 \rho X_2^\dagger + p_3 X_3 \rho X_3^\dagger,$$

where $X_1 = \sigma_x \otimes I_2 \otimes I_2$, etc.



QECC without syndrome measurement.



QECC with syndrome measurement.

Knill-Laflamme Condition

Knill-Laflamme condition

Consider the quantum channel $\mathcal{E} : M_n \to M_n$

$$\mathcal{E}(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger \quad \text{with } F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n.$$

Knill-Laflamme Condition

Knill-Laflamme condition

Consider the quantum channel $\mathcal{E} : M_n \to M_n$

$$\mathcal{E}(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger \quad \text{with } F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n.$$

Then a k-dimensional subspace V of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ satisfying

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \text{whenever } P_V \rho P_V = \rho$$

Knill-Laflamme Condition

Knill-Laflamme condition

Consider the quantum channel $\mathcal{E} : M_n \to M_n$

$$\mathcal{E}(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger \quad \text{with } F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n.$$

Then a k-dimensional subspace V of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ satisfying

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \text{whenever } P_V \rho P_V = \rho$$

for some quantum operation $\mathcal{R}$ if and only if

$$P_V F_i^\dagger F_j P_V = \gamma_{ij} P_V \text{ with } \gamma_{ij} \in \mathbb{C}, 1 \leq i, j \leq r,$$

Knill-Laflamme Condition

Knill-Laflamme condition

Consider the quantum channel $\mathcal{E} : M_n \to M_n$

$$\mathcal{E}(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger \quad \text{with } F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n.$$

Then a k-dimensional subspace V of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ satisfying

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \text{whenever } P_V \rho P_V = \rho$$

for some quantum operation $\mathcal{R}$ if and only if

$$P_V F_i^\dagger F_j P_V = \gamma_{ij} P_V \text{ with } \gamma_{ij} \in \mathbb{C}, 1 \le i, j \le r,$$

where $P_V : \mathbb{C}^n \to V$ is the orthogonal projection such that $P_V(\mathbb{C}^n) = V$.

Knill-Laflamme Condition

Knill-Laflamme condition

Consider the quantum channel $\mathcal{E} : M_n \to M_n$

$$\mathcal{E}(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger \quad \text{with } F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n.$$

Then a k-dimensional subspace V of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ satisfying

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \text{whenever } P_V \rho P_V = \rho$$

for some quantum operation $\mathcal{R}$ if and only if

$$P_V F_i^\dagger F_j P_V = \gamma_{ij} P_V \text{ with } \gamma_{ij} \in \mathbb{C}, 1 \le i, j \le r,$$

where $P_V : \mathbb{C}^n \to V$ is the orthogonal projection such that $P_V(\mathbb{C}^n) = V$.

Remarks

- Suppose one has to protect information in k-qubits. Determine the minimum number of additional qubits needed to protect the information.

Knill-Laflamme condition

Consider the quantum channel $\mathcal{E} : M_n \to M_n$

$$\mathcal{E}(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger \quad \text{with } F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n.$$

Then a k-dimensional subspace V of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ satisfying

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \text{whenever } P_V \rho P_V = \rho$$

for some quantum operation $\mathcal{R}$ if and only if

$$P_V F_i^\dagger F_j P_V = \gamma_{ij} P_V \text{ with } \gamma_{ij} \in \mathbb{C}, 1 \leq i, j \leq r,$$

where $P_V : \mathbb{C}^n \to V$ is the orthogonal projection such that $P_V(\mathbb{C}^n) = V$.

Remarks

- Suppose one has to protect information in k-qubits. Determine the minimum number of additional qubits needed to protect the information. Here, we assume the error pattern will extend to n-qubits for $n \geq k$.

Knill-Laflamme Condition

Knill-Laflamme condition

Consider the quantum channel $\mathcal{E} : M_n \to M_n$

$$\mathcal{E}(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger \quad \text{with } F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n.$$

Then a k-dimensional subspace V of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ satisfying

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \text{whenever } P_V \rho P_V = \rho$$

for some quantum operation $\mathcal{R}$ if and only if

$$P_V F_i^\dagger F_j P_V = \gamma_{ij} P_V \text{ with } \gamma_{ij} \in \mathbb{C}, 1 \le i, j \le r,$$

where $P_V : \mathbb{C}^n \to V$ is the orthogonal projection such that $P_V(\mathbb{C}^n) = V$.

Remarks

- Suppose one has to protect information in k-qubits. Determine the minimum number of additional qubits needed to protect the information. Here, we assume the error pattern will extend to n-qubits for $n \ge k$.
- Find a QECC with maximum dimension for a given channel $\mathcal{E}$.

Finding effective encoding and decoding schemes

Question Syndrome measurement or no syndrome measurement?

Example. For the bit-flip QECC, one can construct

$$\mathcal{R}(\rho) = P\rho P + (I_8 - P)\rho(I_8 - P),$$

where P is the orthogonal projection onto the subspace span$\{|000\rangle, |111\rangle\}$.

Finding effective encoding and decoding schemes

Question Syndrome measurement or no syndrome measurement?

Example. For the bit-flip QECC, one can construct

$$\mathcal{R}(\rho) = P\rho P + (I_8 - P)\rho(I_8 - P),$$

where P is the orthogonal projection onto the subspace span$\{|000\rangle, |111\rangle\}$.

But it is difficult to construct/simulate such a recovery channel!

Finding effective encoding and decoding schemes

Question Syndrome measurement or no syndrome measurement?

Example. For the bit-flip QECC, one can construct

$$\mathcal{R}(\rho) = P\rho P + (I_8 - P)\rho(I_8 - P),$$

where P is the orthogonal projection onto the subspace span$\{|000\rangle, |111\rangle\}$.

But it is difficult to construct/simulate such a recovery channel!

Theorem [Nayak and Zen, 2006],[Kribs and Spekkens,2006],[LNPST,2011]

Suppose a quantum channel $\mathcal{E} : M_n \to M_n$ is given with error operators $F_1, \ldots, F_r$, and there is a k-dimensional QECC V such that $n = mk$.

Finding effective encoding and decoding schemes

Question Syndrome measurement or no syndrome measurement?

Example. For the bit-flip QECC, one can construct

$$\mathcal{R}(\rho) = P\rho P + (I_8 - P)\rho(I_8 - P),$$

where P is the orthogonal projection onto the subspace span$\{|000\rangle, |111\rangle\}$.

But it is difficult to construct/simulate such a recovery channel!

Theorem [Nayak and Zen, 2006],[Kribs and Spekkens,2006],[LNPST,2011]

Suppose a quantum channel $\mathcal{E} : M_n \to M_n$ is given with error operators $F_1, \dots, F_r$, and there is a k-dimensional QECC V such that $n = mk$.

- There is a unitary $U, V \in M_n$ such that

$$\mathcal{E}(U(\tilde{\rho} \oplus O_{n-k})U^\dagger) = V(\xi_{\tilde{\rho}} \otimes \tilde{\rho})V^\dagger,$$

  where $\xi_{\tilde{\rho}} \in M_m$ depending on $\tilde{\rho}$.

Finding effective encoding and decoding schemes

Question Syndrome measurement or no syndrome measurement?

Example. For the bit-flip QECC, one can construct

$$\mathcal{R}(\rho) = P\rho P + (I_8 - P)\rho(I_8 - P),$$

where P is the orthogonal projection onto the subspace span$\{|000\rangle, |111\rangle\}$.

But it is difficult to construct/simulate such a recovery channel!

Theorem [Nayak and Zen, 2006],[Kribs and Spekkens,2006],[LNPST,2011]

Suppose a quantum channel $\mathcal{E} : M_n \to M_n$ is given with error operators $F_1, \ldots, F_r$, and there is a k-dimensional QECC V such that $n = mk$.

- There is a unitary $U, V \in M_n$ such that

$$\mathcal{E}(U(\tilde{\rho} \oplus O_{n-k})U^\dagger) = V(\xi_{\tilde{\rho}} \otimes \tilde{\rho})V^\dagger,$$

  where $\xi_{\tilde{\rho}} \in M_m$ depending on $\tilde{\rho}$. Consequently, the decoding can be done by a unitary similarity transform followed by a partial trace.

- Suppose $\tilde{\mathcal{E}}$ is another quantum channel with error operators lying in the linear span of $\{F_1, \ldots, F_r\}$.

Finding effective encoding and decoding schemes

Question Syndrome measurement or no syndrome measurement?

Example. For the bit-flip QECC, one can construct

$$\mathcal{R}(\rho) = P\rho P + (I_8 - P)\rho(I_8 - P),$$

where P is the orthogonal projection onto the subspace span$\{|000\rangle, |111\rangle\}$.

But it is difficult to construct/simulate such a recovery channel!

Theorem [Nayak and Zen, 2006],[Kribs and Spekkens,2006],[LNPST,2011]

Suppose a quantum channel $\mathcal{E} : M_n \to M_n$ is given with error operators $F_1, \dots, F_r$, and there is a k-dimensional QECC V such that $n = mk$.

- There is a unitary $U, V \in M_n$ such that

  $$\mathcal{E}(U(\tilde{\rho} \oplus O_{n-k})U^\dagger) = V(\xi_{\tilde{\rho}} \otimes \tilde{\rho})V^\dagger,$$

  where $\xi_{\tilde{\rho}} \in M_m$ depending on $\tilde{\rho}$. Consequently, the decoding can be done by a unitary similarity transform followed by a partial trace.

- Suppose $\tilde{\mathcal{E}}$ is another quantum channel with error operators lying in the linear span of $\{F_1, \dots, F_r\}$.

  Then the same encoding / decoding schemes of $\mathcal{E}$ works also for $\tilde{\mathcal{E}}$.

For the bit-flip channel, we can let $U = V$ be a permutation matrix:

$$E_{11} + E_{27} + E_{35} + E_{44} + E_{53} + E_{66} + E_{78} + E_{82}.$$

For the bit-flip channel, we can let $U = V$ be a permutation matrix:

$$E_{11} + E_{27} + E_{35} + E_{44} + E_{53} + E_{66} + E_{78} + E_{82}.$$

The same QECC works for the quantum channel

$$\tilde{\mathcal{E}}(\rho) = p_0\rho + p_1 G_1 \rho G_1^\dagger + p_2 G_2 \rho G_2^\dagger + p_3 G_3 \rho G_3^\dagger,$$

where $G_1 = e^{it_1\sigma_x} \otimes I_2 \otimes I_2$, $G_2 = I_2 \otimes e^{it_2\sigma_x} \otimes I_2$, $G_3 = I_2 \otimes I_2 \otimes e^{it_3\sigma_x}$.

For the bit-flip channel, we can let $U = V$ be a permutation matrix:

$$E_{11} + E_{27} + E_{35} + E_{44} + E_{53} + E_{66} + E_{78} + E_{82}.$$

The same QECC works for the quantum channel

$$\tilde{\mathcal{E}}(\rho) = p_0\rho + p_1 G_1 \rho G_1^\dagger + p_2 G_2 \rho G_2^\dagger + p_3 G_3 \rho G_3^\dagger,$$

where $G_1 = e^{it_1\sigma_x} \otimes I_2 \otimes I_2$, $G_2 = I_2 \otimes e^{it_2\sigma_x} \otimes I_2$, $G_3 = I_2 \otimes I_2 \otimes e^{it_3\sigma_x}$.

A. Nayak and P. Zen, Quantum Inf. & Comp, 2006.

D.W. Kribs and R. Spekkens, Quantum error correcting subsystems are unitarily recoverable subsystems, Physical Review A, 74, 042329 (2006).

C.K. Li, M. Nakahara, Y.T. Poon, N.S. Sze, H. Tomita, Recovery in quantum error correction for general noise without measurement, Quantum Information & Computation 12 (2012), 149-158.

Collective noise

We consider quantum channels in which all physical qubits involved in coding suffer from the same error operators.

Collective noise

We consider quantum channels in which all physical qubits involved in coding suffer from the same error operators.

There are two relevant cases in which such error operators are in action:

Collective noise

We consider quantum channels in which all physical qubits involved in coding suffer from the same error operators.

There are two relevant cases in which such error operators are in action:

- when the size of a quantum computer is much smaller than the wavelength of the external disturbances and

Collective noise

We consider quantum channels in which all physical qubits involved in coding suffer from the same error operators.

There are two relevant cases in which such error operators are in action:

- when the size of a quantum computer is much smaller than the wavelength of the external disturbances and
- when photonic qubits are sent one by one through an optical fiber with a fixed imperfection. In both cases, the qubits suffer from the same errors, leading to decoherence.

Collective noise

We consider quantum channels in which all physical qubits involved in coding suffer from the same error operators.

There are two relevant cases in which such error operators are in action:

- when the size of a quantum computer is much smaller than the wavelength of the external disturbances and
- when photonic qubits are sent one by one through an optical fiber with a fixed imperfection. In both cases, the qubits suffer from the same errors, leading to decoherence.

Another instance in which such encoding is useful is when Alice sends quantum information to Bob (possibly billions of light years away) without knowing which basis vectors Bob employs.

Collective noise

We consider quantum channels in which all physical qubits involved in coding suffer from the same error operators.

There are two relevant cases in which such error operators are in action:

- when the size of a quantum computer is much smaller than the wavelength of the external disturbances and
- when photonic qubits are sent one by one through an optical fiber with a fixed imperfection. In both cases, the qubits suffer from the same errors, leading to decoherence.

Another instance in which such encoding is useful is when Alice sends quantum information to Bob (possibly billions of light years away) without knowing which basis vectors Bob employs.

Then mismatching of the basis vectors is common for all qubits and such mismatching is regarded as collective noise.

Example 1

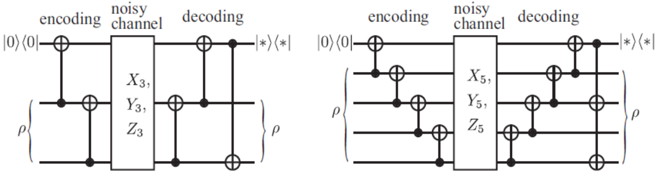
Consider channels with error operators from the set $\{\sigma_x^{\otimes n}, \sigma_y^{\otimes n}, \sigma_z^{\otimes n}\}$.
We implement an recursive encoding/decoding circuits, which protects

$$n-1/n-2 \text{ logical qubits when } n \text{ is odd / even.}$$

Example 1

Consider channels with error operators from the set $\{\sigma_x^{\otimes n}, \sigma_y^{\otimes n}, \sigma_z^{\otimes n}\}$. We implement an recursive encoding/decoding circuits, which protects

$$n-1/n-2 \text{ logical qubits when } n \text{ is odd / even.}$$

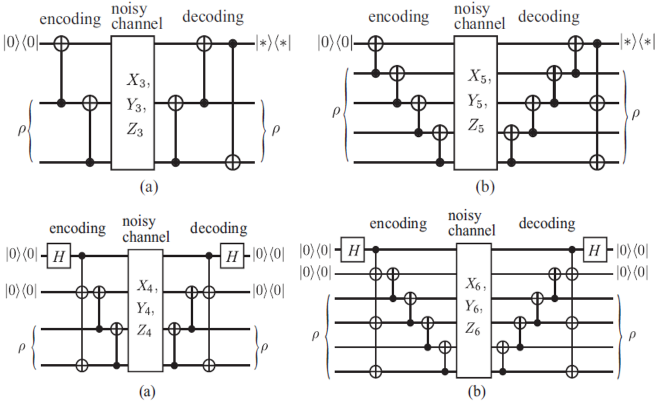


(a) (b)

Example 1

Consider channels with error operators from the set $\{\sigma_x^{\otimes n}, \sigma_y^{\otimes n}, \sigma_z^{\otimes n}\}$. We implement an recursive encoding/decoding circuits, which protects

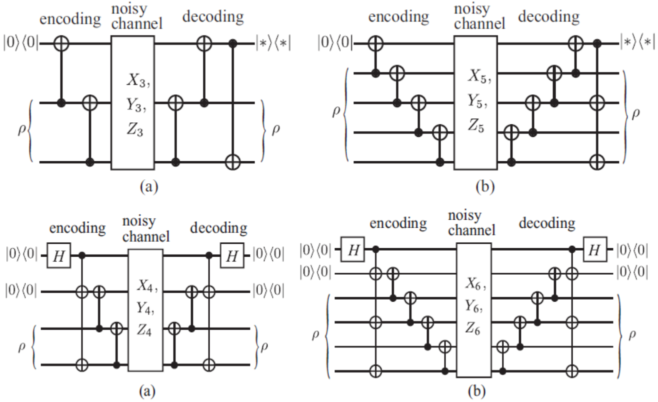
$$n-1/n-2 \text{ logical qubits when } n \text{ is odd / even.}$$



Example 1

Consider channels with error operators from the set $\{\sigma_x^{\otimes n}, \sigma_y^{\otimes n}, \sigma_z^{\otimes n}\}$. We implement an recursive encoding/decoding circuits, which protects

$$n-1/n-2 \text{ logical qubits when } n \text{ is odd / even.}$$



C.K. Li, M. Nakahara, Y.T. Poon, N.S. Sze, H. Tomita, Efficient Quantum Error Correction for Fully Correlated Noise, Phys. Lett. A, 375:3255-3258 (2011).

Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

We gave explicit recursive implementation of encoding/decoding circuits.

Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

We gave explicit recursive implementation of encoding/decoding circuits.

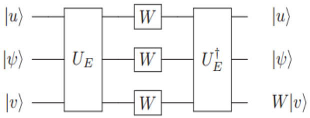
We used $n = 2k + 1$ physical qubits protect k logical qubits.

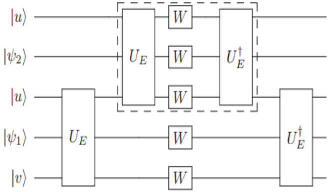
Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

We gave explicit recursive implementation of encoding/decoding circuits.

We used $n = 2k + 1$ physical qubits protect k logical qubits.



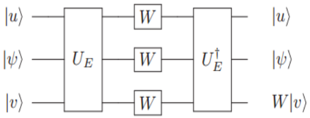


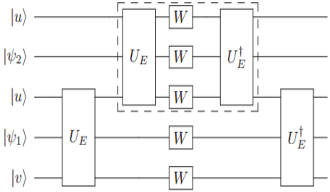
Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

We gave explicit recursive implementation of encoding/decoding circuits.

We used $n = 2k + 1$ physical qubits protect k logical qubits.





Remarks

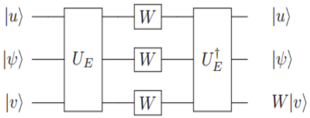
- Note that in the above encoding, $|u\rangle = |0\rangle$ and $|v\rangle$ is arbitrary.

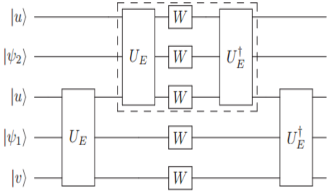
Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

We gave explicit recursive implementation of encoding/decoding circuits.

We used $n = 2k + 1$ physical qubits protect k logical qubits.





Remarks

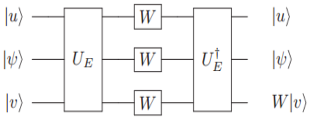
- Note that in the above encoding, $|u\rangle = |0\rangle$ and $|v\rangle$ is arbitrary.
- One can actually find a QECC of higher dimension correcting $f(n)$ qubits such that $f(n)/n \to 1$ as $n \to \infty$.

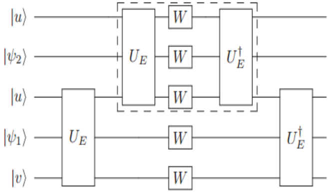
Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

We gave explicit recursive implementation of encoding/decoding circuits.

We used $n = 2k + 1$ physical qubits protect k logical qubits.





Remarks

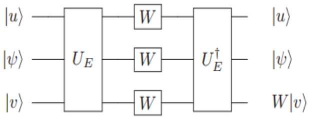
- Note that in the above encoding, $|u\rangle = |0\rangle$ and $|v\rangle$ is arbitrary.
- One can actually find a QECC of higher dimension correcting $f(n)$ qubits such that $f(n)/n \to 1$ as $n \to \infty$.
- The recursive scheme may still be useful because of its efficiency in encoding and decoding.

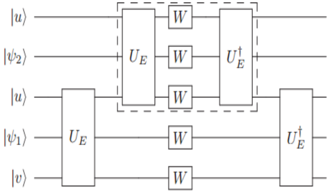
Example 2

Consider channels with error operators of the form $W^{\otimes n}$, where $W \in SU(2)$.

We gave explicit recursive implementation of encoding/decoding circuits.

We used $n = 2k + 1$ physical qubits protect k logical qubits.





Remarks

- Note that in the above encoding, $|u\rangle = |0\rangle$ and $|v\rangle$ is arbitrary.
- One can actually find a QECC of higher dimension correcting $f(n)$ qubits such that $f(n)/n \to 1$ as $n \to \infty$.
- The recursive scheme may still be useful because of its efficiency in encoding and decoding.

C.K. Li, M. Nakahara, Y.T. Poon, N.S. Sze, H. Tomita, Recursive Encoding and Decoding of Noiseless Subsystem and Decoherence Free Subspace, Physical Review A 84, 044301 (2011).

Extension to qudits

- Consider quantum channels on n-qudits (d-dimensional) with error operators of the form $W^{\otimes n}$, where $W \in SU(d)$.

Extension to qudits

- Consider quantum channels on n-qudits (d-dimensional) with error operators of the form $W^{\otimes n}$, where $W \in SU(d)$.
- When $d = 2$, we can use 2 ancillas to protect 1 data qubit.

Extension to qudits

- Consider quantum channels on n-qudits (d-dimensional) with error operators of the form $W^{\otimes n}$, where $W \in SU(d)$.
- When $d = 2$, we can use 2 ancillas to protect 1 data qubit.
- By a suitable set up of the 3 qubit case, we can recursively adding 2 qubits in each step so that 1 of them will be a data qubit.

Extension to qudits

- Consider quantum channels on n-qudits (d-dimensional) with error operators of the form $W^{\otimes n}$, where $W \in SU(d)$.
- When $d = 2$, we can use 2 ancillas to protect 1 data qubit.
- By a suitable set up of the 3 qubit case, we can recursively adding 2 qubits in each step so that 1 of them will be a data qubit.
- For qutrit case, one can use 3 ancillas to protect 1 data qutrit.

Extension to qudits

- Consider quantum channels on n-qudits (d-dimensional) with error operators of the form $W^{\otimes n}$, where $W \in SU(d)$.
- When $d = 2$, we can use 2 ancillas to protect 1 data qubit.
- By a suitable set up of the 3 qubit case, we can recursively adding 2 qubits in each step so that 1 of them will be a data qubit.
- For qutrit case, one can use 3 ancillas to protect 1 data qutrit.
- By a suitable set up of the 4 qutrit case, we can recursively adding 3 qutrits in each step so that 1 of them will be a data qutrit.

Theorem [GLNPS]

For the n-qudit channels described above, we can use $n = dk + 1$ physical qudits protect k logical qubits.

Extension to qudits

- Consider quantum channels on n-qudits (d-dimensional) with error operators of the form $W^{\otimes n}$, where $W \in SU(d)$.
- When $d = 2$, we can use 2 ancillas to protect 1 data qubit.
- By a suitable set up of the 3 qubit case, we can recursively adding 2 qubits in each step so that 1 of them will be a data qubit.
- For qutrit case, one can use 3 ancillas to protect 1 data qutrit.
- By a suitable set up of the 4 qutrit case, we can recursively adding 3 qutrits in each step so that 1 of them will be a data qutrit.

Theorem [GLNPS]

For the n-qudit channels described above, we can use $n = dk + 1$ physical qudits protect k logical qubits. So, the "error correction rate" is $k/(dk+1) \to 1/d$ as $n = dk + 1 \to \infty$.

Extension to qudits

- Consider quantum channels on n-qudits (d-dimensional) with error operators of the form $W^{\otimes n}$, where $W \in SU(d)$.
- When $d = 2$, we can use 2 ancillas to protect 1 data qubit.
- By a suitable set up of the 3 qubit case, we can recursively adding 2 qubits in each step so that 1 of them will be a data qubit.
- For qutrit case, one can use 3 ancillas to protect 1 data qutrit.
- By a suitable set up of the 4 qutrit case, we can recursively adding 3 qutrits in each step so that 1 of them will be a data qutrit.

Theorem [GLNPS]

For the n-qudit channels described above, we can use $n = dk + 1$ physical qudits protect k logical qubits. So, the "error correction rate" is $k/(dk+1) \to 1/d$ as $n = dk + 1 \to \infty$.

Remark One can actually find a QECC of higher dimension correcting $f(n)$ qudits such that $f(n)/n \to 1$ as $n \to \infty$.

Idea of proofs and related questions

Idea of proofs and related questions

- Find the irreducible representations of the Lie algebra generated by the error operators, and write it as a direct sum of

$$I_{f_j} \otimes M_{g_j}, \quad j = 1, \ldots, \ell.$$

Idea of proofs and related questions

- Find the irreducible representations of the Lie algebra generated by the error operators, and write it as a direct sum of

$$I_{f_j} \otimes M_{g_j}, \quad j = 1, \dots, \ell.$$

- The dimension of the irreducible representations in the decomposition are determined by Clebsch-Gordan coefficients.

Idea of proofs and related questions

- Find the irreducible representations of the Lie algebra generated by the error operators, and write it as a direct sum of

$$I_{f_j} \otimes M_{g_j}, \quad j = 1, \ldots, \ell.$$

- The dimension of the irreducible representations in the decomposition are determined by Clebsch-Gordan coefficients.
- Choose a factor $I_{f_j} \otimes M_{g_j}$ containing vectors with a lot of symmetries so that we can do the recursive scheme.

Idea of proofs and related questions

- Find the irreducible representations of the Lie algebra generated by the error operators, and write it as a direct sum of

$$I_{f_j} \otimes M_{g_j}, \quad j = 1, \dots, \ell.$$

- The dimension of the irreducible representations in the decomposition are determined by Clebsch-Gordan coefficients.
- Choose a factor $I_{f_j} \otimes M_{g_j}$ containing vectors with a lot of symmetries so that we can do the recursive scheme.
- In fact, one can encode $\tilde{\rho}$ as $\rho = \sigma \otimes \tilde{\rho} \oplus O_{N-f_j q_j}$ so that the channel will take ρ to $\hat{\sigma} \otimes \tilde{\rho} \oplus O$.

Idea of proofs and related questions

- Find the irreducible representations of the Lie algebra generated by the error operators, and write it as a direct sum of

$$I_{f_j} \otimes M_{g_j}, \quad j = 1, \dots, \ell.$$

- The dimension of the irreducible representations in the decomposition are determined by Clebsch-Gordan coefficients.
- Choose a factor $I_{f_j} \otimes M_{g_j}$ containing vectors with a lot of symmetries so that we can do the recursive scheme.
- In fact, one can encode $\tilde{\rho}$ as $\rho = \sigma \otimes \tilde{\rho} \oplus O_{N-f_j q_j}$ so that the channel will take ρ to $\hat{\sigma} \otimes \tilde{\rho} \oplus O$.
- Thus, $\tilde{\rho}$ lies in a noiseless subsystem, and one can correct/recover $\tilde{\rho}$.

Idea of proofs and related questions

- Find the irreducible representations of the Lie algebra generated by the error operators, and write it as a direct sum of

$$I_{f_j} \otimes M_{g_j}, \quad j = 1, \ldots, \ell.$$

- The dimension of the irreducible representations in the decomposition are determined by Clebsch-Gordan coefficients.
- Choose a factor $I_{f_j} \otimes M_{g_j}$ containing vectors with a lot of symmetries so that we can do the recursive scheme.
- In fact, one can encode $\tilde{\rho}$ as $\rho = \sigma \otimes \tilde{\rho} \oplus O_{N-f_j q_j}$ so that the channel will take ρ to $\hat{\sigma} \otimes \tilde{\rho} \oplus O$.
- Thus, $\tilde{\rho}$ lies in a noiseless subsystem, and one can correct/recover $\tilde{\rho}$.

U. Gungordu, C.K. Li, M. Nakahara, Y.T. Poon and N.K. Sze, Recursive encoding and decoding of the noiseless subsystem for qudits, Physical Review A, to appear.

D. Kribs and W. Seppkens, Quantum Error Correcting Subsystems are Unitarily Recoverable Subsystems, Physical Review A, 74, 042329 (2006).

Using mixed ancillas for encoding

- Nakahara and his student found out that in the quantum channel with error operators in $\{\otimes^3\sigma_x, \otimes^3\sigma_y, \otimes^3\sigma_z\}$, one can use two arbitrary ancillas to protect one data qubit.

Using mixed ancillas for encoding

- Nakahara and his student found out that in the quantum channel with error operators in $\{\otimes^3 \sigma_x, \otimes^3 \sigma_y, \otimes^3 \sigma_z\}$, one can use two arbitrary ancillas to protect one data qubit.
- One can use 1 pure ancilla to protect 2 data qubits, but the above scheme is attractive in NMR quantum computing as preparing pure ancillas is expensive.

Using mixed ancillas for encoding

- Nakahara and his student found out that in the quantum channel with error operators in $\{\otimes^3 \sigma_x, \otimes^3 \sigma_y, \otimes^3 \sigma_z\}$, one can use two arbitrary ancillas to protect one data qubit.
- One can use 1 pure ancilla to protect 2 data qubits, but the above scheme is attractive in NMR quantum computing as preparing pure ancillas is expensive.

Proposition [GLNPS]

Given quantum channel $\mathcal{E}$ on n-qubits with error operators $F_1, \ldots, F_r$. One can use p pure ancillas and q mixed ancillas to protect $k = n - p - q$ data qubits if and only if

Using mixed ancillas for encoding

- Nakahara and his student found out that in the quantum channel with error operators in $\{\otimes^3 \sigma_x, \otimes^3 \sigma_y, \otimes^3 \sigma_z\}$, one can use two arbitrary ancillas to protect one data qubit.
- One can use 1 pure ancilla to protect 2 data qubits, but the above scheme is attractive in NMR quantum computing as preparing pure ancillas is expensive.

Proposition [GLNPS]

Given quantum channel $\mathcal{E}$ on n-qubits with error operators $F_1, \dots, F_r$. One can use p pure ancillas and q mixed ancillas to protect $k = n - p - q$ data qubits if and only if there is a partial isometry X such that

$$X^\dagger F_i^\dagger F_j X = B_{ij} \otimes I_{2^k} \text{ with } B_{ij} \in M_{2^q}, 1 \leq i, j \leq r.$$

Corollary

- For the quantum channel with error operators in $\{\otimes^n \sigma_x, \otimes^n \sigma_y, \otimes^n \sigma_z\}$, one can use arbitrary ancilla(s) to protect $n-1/n-2$ data bits if n is odd/even.

Corollary

- For the quantum channel with error operators in $\{\otimes^n \sigma_x, \otimes^n \sigma_y, \otimes^n \sigma_z\}$, one can use arbitrary ancilla(s) to protect $n-1/n-2$ data bits if n is odd/even.
- For the bit-flip channel on M_8, one has to use 2 pure ancillas to protect 1 data qubit.

Open problems

Can one use 1 pure ancilla, ℓ arbitrary ancillas to protect one data qubit?

Corollary

- For the quantum channel with error operators in $\{\otimes^n \sigma_x, \otimes^n \sigma_y, \otimes^n \sigma_z\}$, one can use arbitrary ancilla(s) to protect $n-1/n-2$ data bits if n is odd/even.
- For the bit-flip channel on M_8, one has to use 2 pure ancillas to protect 1 data qubit.

Open problems

Can one use 1 pure ancilla, ℓ arbitrary ancillas to protect one data qubit?

One may ask similar questions for other channels, and further research is needed.

Thank you for your attention!