Math 410 Final Examination **Due: 11:59 p.m., 17 December, 2018**

If you have time and interest, try 2(c), 3(c), 8(d) for extra credits. Good Luck!

1. Suppose $A \in M_m$ and $B \in M_n$ have eigenvalues $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$, respectively.
   (a) Show that $A \otimes B$ has eigenvalues $a_i b_j$ with $1 \le i \le m, 1 \le j \le n$.
   (b) Determine the eigenvalues and their multiplicities of each of the following matrices
   $\sigma_z \otimes \cdots \otimes \sigma_z$ (n times), $\sigma_x \otimes \cdots \otimes \sigma_x$ (n times), $H \otimes \cdots \otimes H$ (n times),
   where $H \in M_2$ is the Hadamard gate.

2. Let $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ and $\sigma = \begin{pmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{pmatrix} \in M_2$ be density matrices. Suppose λ_1, λ_2 are the eigenvalues of $\rho^{1/2}\sigma\rho^{1/2}$.
   (a) Show that $\lambda_1 + \lambda_2 = pa_{11} + (1-p)a_{22}$ and $\lambda_1\lambda_2 = p(1-p)\det(\sigma)$.
   (b) Recall that the fidelity of ρ, σ is $F(\rho,\sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = \lambda_1^{1/2} + \lambda_2^{1/2}$, Show that
   $$F(\rho,\sigma)^2 \ge pa_{11} + (1-p)a_{22} \ge \min\{p, 1-p\}.$$
   (c) (Extra 5 points) Determine ALL the density matrices $\sigma \in M_2$ such that the fidelity $F(\rho,\sigma) = \sqrt{p}$ if $p \le 1/2$. [Hint: Consider 2 cases: $p \ne 1/2$, $p = 1/2$.]

3. (See §9.1 and §9.2) Let $F_1, \ldots, F_r$ be $m \times n$ complex matrices. Define $\Phi : M_n \to M_m$ by
   $$\Phi(A) = F_1 A F_1^\dagger + \cdots + F_r A F_r^\dagger.$$
   (a) Show that if A is positive semidefinite, then $\Phi(A)$ is positive semidefinite. That is, show that if $\langle v|A|v\rangle \ge 0$ for all $|v\rangle \in \mathbb{C}^n$, then $\langle u|\Phi(A)|u\rangle \ge 0$ for all $|u\rangle \in \mathbb{C}^m$. [Hint: Show that $\langle u|F_j A F_j^\dagger|u\rangle \ge 0$ for every j.]
   (b) Show that if $F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n$, then $\mathrm{Tr}(A) = \mathrm{Tr}(\Phi(A))$ for all $A \in M_n$. [Hint: Note that $\mathrm{Tr}(F_j A F_j^\dagger) = \mathrm{Tr}(A F_j^\dagger F_j)$ for every j.]
   (c) (Extra 5 points) Show that if $\mathrm{Tr}(A) = \mathrm{Tr}(\Phi(A))$ for all density matrices $A \in M_n$, then $F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n$.

4. Determine the change of the Bloch sphere under the following quantum operation:
   $$\mathcal{E}(\rho) = (1-2p)\rho + p(\sigma_x\rho\sigma_x + \sigma_z\rho\sigma_z), \qquad p \in (0, 1/2).$$

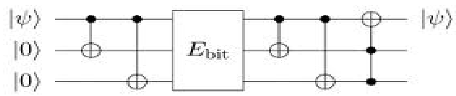
5. In the nine-qubit code (§10.3), determine the error and the correction operation in each of the following.
   (a) $(A_i, B_i) = (0,0)$ for $i = 1,2,3$ and $(A_4, B_4) = (1,0)$.
   (b) $(A_i, B_i) = (0,0)$ for $i = 1,2$, and $(A_j, B_j) = (0,1)$ for $j = 3,4$.

6. For the five-qubit QECC described in §10.5.2, show that if $|\Psi\rangle$ is changed to $Y_0|\Psi\rangle$, then

$$M_0|\Psi\rangle = |\Psi\rangle \quad \text{and} \quad M_i|\Psi\rangle = -|\Psi\rangle \quad \text{for } i = 1, 2, 3.$$

Remark. The point of this problem is to verify the seventh column of Table 10.70. Using the table to "prove" the result is not acceptable.

7. Consider the following encoding / decoding circuit for the three qubit bit-flip quantum error correction scheme:



Suppose $|\psi\rangle = a|0\rangle + b|1\rangle$ goes through the circuit. Determine the final three qubit sate if

(a) an error occurs in the first qubibit,

(b) an error occurs in the second qubit,

(c) an error occurs in the third qubit.

8. Suppose one can encode 2-qubit states as n-qubit states so that 2-bit general errors can be corrected, i.e., up to 2-bit errors caused by X, Y, Z can be corrected. In particular, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are encoded as the logical n-qubit states: $|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L$.

(a) Show that there should be $1 + 3n + 3^2n(n-1)/2$ states in $\{|x_1 \cdots x_n\rangle : x_j \in \{0, 1\}\}$ decoded as $|00\rangle_L$.

(b) Show that $4(1 + 3n + 9n(n-1)/2) \leq 2^n$.

(c) Find the smallest $n \in \mathbb{N}$ satisfying the condition in (b).

(d) (Extra 20 points) Prove or disprove that one can construct an n-qubit QECC for general errors with the minimum n found in the above problem.