Math 410 Final Examination **Sample solution.**

If you have time and interest, try 2(c), 3(c), 8(d) for extra credits. Good Luck!

1. Suppose $A \in M_m$ and $B \in M_n$ have eigenvalues $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$, respectively.
   (a) Show that $A \otimes B$ has eigenvalues $a_i b_j$ with $1 \le i \le m, 1 \le j \le n$.
   (b) Determine the eigenvalues and their multiplicities of each of the following matrices
   $\sigma_z \otimes \cdots \otimes \sigma_z$ (n times), $\sigma_x \otimes \cdots \otimes \sigma_x$ (n times), $H \otimes \cdots \otimes H$ (n times),
   where $H \in M_2$ is the Hadamard gate.

   Solution. (a) Note that there are unitary U and V such that $U^\dagger A U = T_1$ and $V^\dagger B V = T_2$, where T_1 and T_2 are triangular matrices with diagonal entries $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$, respectively. Thus, $(U \otimes V)^\dagger (A \otimes B)(U \otimes V) = T_1 \otimes T_2$ is in triangular form with diagonal entries $a_i b_j$, $1 \le i \le m, 1 \le j \le n$. We conclude that $A \otimes B$ has the asserted eigenvalues with desired multiplicity.

   [Note: Here we do not need to assume that A and B have complete sets of linearly independent eigenvectors, i.e., A and B are diagonalizable. Also, if one use the fact that $A|a\rangle = a|a\rangle$ and $B|b\rangle = b|b\rangle$ so that $(A \otimes B)|a \otimes b\rangle = (ab)|a \otimes b\rangle$, one can only conclude that every pair of eigenvalues a, b of A and B yields an eigenvalue ab of $A \otimes B$, but it does not yield the multiplicities of the eigenvalues.]

   (b) Each $A \in \{\sigma_x, \sigma_z, H\}$ has eigenvalues $1, -1$. We claim $A \otimes \cdots \otimes A$ ($n\times$) has eigenvalues $1, -1$, both have multiplicities 2^{n-1}. The result holds for $n = 1$. Assume that the result holds for the $n-1$ fold tensor $A^{\otimes(n-1)}$. Then $A \otimes (A^{\otimes(n-1)})$ has half of the eigenvalues equal to those of $A^{\otimes(n-1)}$ and the other half of the eigenvalues equal to (-1) times those of of $A^{\otimes(n-1)}$. The result follows.

2. Let $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ and $\sigma = \begin{pmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{pmatrix} \in M_2$ be density matrices. Suppose λ_1, λ_2 are the eigenvalues of $\rho^{1/2}\sigma\rho^{1/2}$.
   (a) Show that $\lambda_1 + \lambda_2 = pa_{11} + (1-p)a_{22}$ and $\lambda_1\lambda_2 = p(1-p)\det(\sigma)$.
   (b) Recall that the fidelity of ρ, σ is $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = \lambda_1^{1/2} + \lambda_2^{1/2}$, Show that
   $$F(\rho, \sigma)^2 \ge pa_{11} + (1-p)a_{22} \ge \min\{p, 1-p\}.$$
   (c) (Extra 5 points) Determine ALL the density matrices $\sigma \in M_2$ such that the fidelity $F(\rho, \sigma) = \sqrt{p}$ if $p \le 1/2$. [Hint: Consider 2 cases: $p \ne 1/2$, $p = 1/2$.]

   Solution. (a) Note that $\rho^{1/2}\sigma\rho^{1/2} = \begin{pmatrix} pa_{11} & \sqrt{p(1-p)}a_{12} \\ \sqrt{p(1-p)}\bar{a}_{12} & (1-p)a_{22} \end{pmatrix}$. So,
   $$\det(tI - \rho^{1/2}\sigma\rho^{1/2}) = t^2 - (pa_{11} + (1-p)a_{22})t + p(1-p)(a_{11}a_{22} - |a_{12}|^2).$$
   By the theory of quadratic equation, the sum of roots equals $pa_{11} + (1-p)a_{22}$ and the product of roots equals $p(1-p)\det(\sigma)$.

(b) Since σ is a density matrix, $a_{11}+a_{22}=1$ and $\det(\sigma)\geq 0$. Thus, if $p\leq 1/2\leq 1-p$, then

$$F(\rho,\sigma)^2=(\lambda_1^{1/2}+\lambda_2^{1/2})^2\geq\lambda_1+\lambda_2=pa_{11}+(1-p)a_{22}\geq\delta(a_{11}+a_{22})=\delta, \tag{1}$$

where $\delta=\min\{p,1-p\}$ and λ_1,λ_2 are eigenvalues of $\rho^{1/2}\sigma\rho^{1/2}$.

(c) If $p=1/2$, then $\rho^{1/2}\sigma\rho^{1/2}=\sigma/2$ and

$$F(\rho,\sigma)=\frac{1}{\sqrt{2}}\mathrm{Tr}\sqrt{\sigma}=1/\sqrt{2}$$

if and only if σ has eigenvalues $1,0$, i.e., σ is a pure state.

If $p<1/2$, then the inequalities in (1) hold if and only if $\lambda_1\lambda_2=0$ and $a_{22}=0$, i.e., σ has rank one and $a_{22}=0$. Equivalently, $\sigma=\mathrm{diag}\,(1,0)$.

3. (See §9.1 and §9.2) Let $F_1,\ldots,F_r$ be $m\times n$ complex matrices. Define $\Phi:M_n\to M_m$ by

$$\Phi(A)=F_1AF_1^\dagger+\cdots+F_rAF_r^\dagger.$$

(a) Show that if A is positive semidefinite, then $\Phi(A)$ is positive semidefinite. That is, show that if $\langle v|A|v\rangle\geq 0$ for all $|v\rangle\in\mathbb{C}^n$, then $\langle u|\Phi(A)|u\rangle\geq 0$ for all $|u\rangle\in\mathbb{C}^m$.

[Hint: Show that $\langle u|F_jAF_j^\dagger|u\rangle\geq 0$ for every j.]

(b) Show that if $F_1^\dagger F_1+\cdots+F_r^\dagger F_r=I_n$, then $\mathrm{Tr}(A)=\mathrm{Tr}(\Phi(A))$ for all $A\in M_n$.

[Hint: Note that $\mathrm{Tr}(F_jAF_j^\dagger)=\mathrm{Tr}(AF_j^\dagger F_j)$ for every j.]

(c) (Extra 5 points) Show that if $\mathrm{Tr}(A)=\mathrm{Tr}(\Phi(A))$ for all density matrices $A\in M_n$, then $F_1^\dagger F_1+\cdots+F_r^\dagger F_r=I_n$.

Solution. (a) Suppose A is positive semidefinite. For each j, for any vector $|v\rangle$, we have $\langle v|F_jAFj^\dagger|v\rangle=\langle u|A|u\rangle\geq 0$, where $|u\rangle=F_j^\dagger|v\rangle$, and thus

$$\langle v|\Phi(A)|v\rangle=\sum_{j=1}^r\langle vF_jAF_j^\dagger|v\rangle\geq 0.$$

Hence, $\Phi(A)$ is positive semidefinite.

(b) If $F_1^\dagger F_1+\cdots+F_r^\dagger F_r=I_n$, then for all $A\in M_n$,

$$\mathrm{Tr}(\Phi(A))=\mathrm{Tr}(\sum_{j=1}^r F_jAF_j^\dagger=\sum_{j=1}^r\mathrm{Tr}(AF_j^\dagger F_j))=\mathrm{Tr}A(\sum_{j=1}^r F_j^\dagger F_j)=\mathrm{Tr}A.$$

(c) Suppose $\mathrm{Tr}A=\mathrm{Tr}\Phi(A)=\mathrm{Tr}(A(\sum_{j=1}F_j^\dagger F_j))$ for all density matrices $A\in M_n$. Let $G=(\sum_{j=1}F_j^\dagger F_j))$. Then G is Hermitian and has eigenvalues $g_1\geq\cdots\geq g_n$. If there is $g_j\neq 1$, then for the pure state $A=|g_j\rangle\langle g_j|$, we have

$$\mathrm{Tr}AG=\mathrm{Tr}|g_j\rangle\langle g_j|G=\langle g_j|G|g_j\rangle=g_j\neq 1=\mathrm{Tr}A.$$

Thus, all eigenvalues of G equals 1 and $G=I_n$.

4. Determine the change of the Bloch sphere under the following quantum operation:

$$\mathcal{E}(\rho) = (1-2p)\rho + p(\sigma_x\rho\sigma_x + \sigma_z\rho\sigma_z), \qquad p \in (0, 1/2).$$

Solution. If $\rho = \frac{1}{2}(I_2 + c_x\sigma_x + c_y\sigma_y + c_z\sigma_z\sigma_z)$, then

$$\mathcal{E}(\rho) = \frac{1}{2}(I_2 + (1-2p)c_x\sigma_x + (1-4p)c_y\sigma_y + (1-2p)c_z\sigma_z\sigma_z).$$

So, $\mathcal{E}$ will change the Bloch sphere to an ellipsoid such that the x-axis and z-axis are scaled by $1-2p \in (0,1)$, and the y-axis is scaled by $1-4p \in (-1/2, 1)$.

Note that points are deducted if you do not describe the effect of $\mathcal{E}$ on the Bloch sphere.

5. In the nine-qubit code (§10.3), determine the error and the correction operation in each of the following.

(a) $(A_i, B_i) = (0,0)$ for $i = 1,2,3$ and $(A_4, B_4) = (1,0)$.

(b) $(A_i, B_i) = (0,0)$ for $i = 1,2$, and $(A_j, B_j) = (0,1)$ for $j = 3,4$.

Solution. (a) There is no bit-flip error in the 9 qubits, and a phase flip error in the second **group** of qubits (4,5, or 6th qubit). We can then apply a bit flip to any one of the 4,5, or 6th qubit to correct the error.

(b) There is a bit flip in the 9th qubit, and also a phase flip in the third group of qubits (7,8,9th qubit). Since, we assume one bit error, we see that the 9th qubit has bit flip and phase flip error. To do the correction, we first apply a bit flip correction to the 9th qubit, and then a phase flip correction to the 9th or any of the 7,8,9th qubit.

Note that points will be deducted if the solution does not show that one can apply phase flip error to any one of the qubit in the group containing the qubit with phase flip error. Also, one cannot just say that we apply a Y error correction to the third group of qubits.

6. For the five-qubit QECC described in §10.5.2, show that if $|\Psi\rangle$ is changed to $Y_0|\Psi\rangle$, then

$$M_0|\Psi\rangle = |\Psi\rangle \quad \text{and} \quad M_i|\Psi\rangle = -|\Psi\rangle \quad \text{for } i = 1,2,3.$$

Remark. The point of this problem is to verify the seventh column of Table 10.70.
Using the table to "prove" the result is not acceptable.
Solution. Note that for the original $|\Psi\rangle$ we have $M_i|\Psi\rangle = |\Psi\rangle$ for $i = 0,1,2,3$. That is $|\Psi\rangle$ is an eigenvector for M_i of the eigenvalue 1 for each $i = 0,1,2,3$. Now,

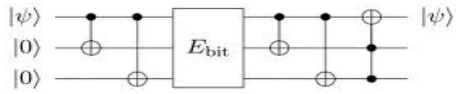
$$Y_0M_0 - M_0Y_0 = Y_0X_2X_3Z_1Z_4 - X_2X_3Z_1Z_4Y_0 = 0,$$
$$Y_0M_1 + M_1Y_0 = Y_0X_3X_4Z_2Z_0 - X_3X_4Z_2Z_0Y_0 = X_3X_4Z_2(X_0Z_0 - Z_0Y_0) = 0,$$
$$Y_0M_2 + M_2Y_0 = Y_0X_4X_0Z_3Z_1 - X_4X_0Z_3Z_1Y_0 = X_4(X_0Y_0 - Y_0X_0)Z_3Z_1 = 0,$$
$$Y_0M_3 + M_3Y_0 = Y_0X_0X_1X_1Z_4Z_2 - X_0X_1Z_4Z_2Y_0 = (X_0Y_0 - Y_0X_0)X_1Z_4Z_2 = 0.$$

It follows that for $|\hat{\Psi}\rangle = Y_0|\Psi\rangle$, $M_0|\hat{\Psi}\rangle = M_0Y_0|\Psi\rangle = Y_0M_0|\Psi\rangle = Y_0|\Psi\rangle = |\hat{\Psi}\rangle$ and

$$M_i|\hat{\Psi}\rangle = M_iY_0|\Psi\rangle = -Y_0M_i|\Psi\rangle = Y_0|\Psi\rangle = -|\hat{\Psi}\rangle \quad \text{for } i = 1,2,3.$$

Note that it is not right to say that $M_0 = I$. Explanation is needed to say that M_0 is "essentially" equal I, etc.

7. Consider the following encoding / decoding circuit for the three qubit bit-flip quantum error correction scheme:



Suppose $|\psi\rangle = a|0\rangle + b|1\rangle$ goes through the circuit. Determine the final three qubit sate if

(a) an error occurs in the first qubibit,

(b) an error occurs in the second qubit,

(c) an error occurs in the third qubit.

Solution. A careful checking of the circuit will show that if $a|000\rangle + b|111\rangle$ is the input state, then the output states in (a), (b), (c) are, respectively,

$$(a|0\rangle + b|1\rangle)|11\rangle, \qquad (a|0\rangle + b|1\rangle)|10\rangle, \qquad (a|0\rangle + b|1\rangle)|01\rangle.$$

8. Suppose one can encode 2-qubit states as n-qubit states so that 2-bit general errors can be corrected, i.e., up to 2-bit errors caused by X, Y, Z can be corrected. In particular, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are encoded as the logical n-qubit states: $|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L$.

(a) Show that there should be $1+3n+3^2n(n-1)/2$ states in $\{|x_1 \cdots x_n\rangle : x_j \in \{0,1\}\}$ decoded as $|00\rangle_L$.

(b) Show that $4(1+3n+9n(n-1)/2) \leq 2^n$.

(c) Find the smallest $n \in \mathbb{N}$ satisfying the condition in (b).

(d) (Extra 20 points) Prove or disprove that one can construct an n-qubit QECC for general errors with the minimum n found in the above problem.

Solution. (a) Suppose $|00\rangle_L$ is an n-qubit state. We will the received state $|x_1 \ldots, x_n\rangle$ as $|00\rangle$ it is the same as $|00\rangle$, or it is the $3n$ states such that one of the bit differs with $|00\rangle$ under an X, Y or Z error, or one of the $\binom{n}{2}3^2$ states that differ in two bits with $|00\rangle$ under two errors from $\{X, Y, Z\}$. So, the total numbers is $1 + 3n + 9n(n-1)/2$.

(b) For a valid QECC, each $|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L$ will associate with set of $1+3n+9n(n-1)/2$ states $|x_1 \ldots x_n\rangle$, and all these sets should be disjoint. Else, one will not be able to decode $|x_1 \ldots x_n\rangle$ if it lies in two of the sets. Hence, there should be at least $4(1+3n+9n(n-1)/2)$ among the 2^n physical states, i.e.,

$$4(1+3n+9n(n-1)/2) \leq 2^n.$$

(c) An easy computer experiment shows that $n = 12$ is the smallest integer n satisfying the inequality.

(d) Open problem for Math 410 in the Spring semester!