Math 410 Introduction to Quantum Computing Homework 9

Sample Solution

8.1 Let $m = 6$.

Step 1 : $\gcd(6, 35) = 1$.

Step 2 : $6^2 = 36 \equiv 1 \pmod{35}$, $P = 2$.

Step 3 : $P = 2$ is even.

Step 4 : $(6^1 - 1)(6^1 + 1) = 35 \equiv 0 \pmod{35}$.

Step 5 : $\gcd((6^1 - 1), 35) = 5, \frac{35}{5} = 7, 35 = 5 \cdot 7$.

8.2 Note that $441 < 2^n < 882$; $n = 9, Q = 2^n = 512$. We can show that $P = 6$ is the period of 11 in $\mathbb{Z}_{21}$. For example, use WolframAlpha to solve equation $Mod[11^a, 21] = 1$. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(a) = m^a$, $|\psi_0\rangle = |REG_1\rangle|REG_2\rangle = |00...0\rangle|00...0\rangle$. Then $|\psi_1\rangle = \frac{1}{\sqrt{Q}} \sum_x |x\rangle|0\rangle$, $|\psi_2\rangle = U_f|\psi_1\rangle = \frac{1}{Q} \sum_x |x\rangle|f(x)\rangle$.

Apply QFT on $|REG_1\rangle$, we yield $|\psi_3\rangle = (\mathcal{F} \otimes I)|\psi_2\rangle = \frac{1}{512} \sum_x \sum_y \omega_n^{-xy} |y\rangle|f(x)\rangle$.

Let $Q = Pq + r, p \in \mathbb{N} \cup \{0\}, 0 \le r < P$, $Q_0 = Pq$. By the formula in p.145,

$$Prob(y) = \begin{cases} \frac{r \sin^2(\frac{\pi P y}{Q}(\frac{Q_0}{P}+1)) + (P-r)\sin^2(\frac{\pi P y}{Q} \cdot \frac{Q_0}{P})}{Q^2 \sin^2(\frac{\pi P y}{Q})} & Py \not\equiv 0 \pmod{Q}, \\ \frac{r(Q_0+P)^2 + (P-r)Q_0^2}{Q^2 P^2} & Py \equiv 0 \pmod{Q}. \end{cases}$$

That is,

$$Prob(y) = \begin{cases} \frac{r \sin^2(\frac{3\pi y}{256}(\frac{Q_0}{6}+1)) + (6-r)\sin^2(\frac{3\pi y}{256} \cdot \frac{Q_0}{6})}{262144 \sin^2(\frac{6\pi y}{512})} & 6y \not\equiv 0 \pmod{512}, \\ \frac{r(Q_0+6)^2 + (6-r)Q_0^2}{9437184} & 6y \equiv 0 \pmod{512}. \end{cases}$$

We see that $Prob(y) \sim 0.167$ if $Py \equiv 0 \pmod{Q}$ and $Prob(y) \sim 0.00087$ if $Py \not\equiv 0 \pmod{Q}$.

8.3 Note that $\frac{61}{45} = 1 + \frac{1}{2+\frac{16}{13}} = 1 + \frac{1}{2+\frac{1}{1+\frac{13}{3}}} = 1 + \frac{1}{2+\frac{1}{1+\frac{1}{4+\frac{1}{3}}}}$. Therefore, $\frac{61}{45} = [1, 2, 1, 4, 3]$.

Note that $\frac{121}{13} = 9 + \frac{1}{\frac{13}{4}} = 9 + \frac{1}{3+\frac{1}{4}}$. Therefore, $\frac{121}{13} = [9, 3, 4]$

8.4 Note that $\frac{37042}{1048576} = [0, 28, 3, 4, 88, 1, 4, 3]$.

Let $p_0 = a_0 = 0, q_0 = 1$. For $p_1 = 0 + 1 = 1; q_1 = 28$, we have $|\frac{1}{28} - \frac{37042}{1048576}| = \frac{1181073}{36700160} > \frac{1}{2Q}$.

For $p_2 = 3 + 0 = 3; q_2 = 3 \cdot 28 + 1 = 85$, we have $|\frac{3}{85} - \frac{37042}{1048576}| = \frac{2830871}{89128960} > \frac{1}{2Q}$.

For $p_3 = 13; q_2 = 368$, we have $|\frac{13}{368} - \frac{37042}{1048576}| = \frac{1}{12058624} < \frac{1}{2Q}$.

Therefore, $P = q_3 = 368$.

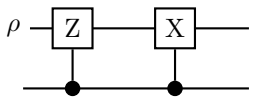
Note that $\frac{65536}{1048576} = [0, 16]$. We have $p_1 = 1, q_1 = 16$ and $|\frac{1}{16} - \frac{65536}{1048576}| = 0 < 1/(2Q)$. So, $P = 16$.

9.1 Note that $(U_a^\dagger \otimes (|\epsilon_a\rangle\langle\epsilon_a|)^\dagger)(U_a \otimes |\epsilon_a\rangle\langle\epsilon_a|) = I \otimes |\epsilon_a\rangle\langle\epsilon_a|$.

If $a \neq b$, then $(U_a^\dagger \otimes (|\epsilon_a\rangle\langle\epsilon_a|)^\dagger)(U_b \otimes |\epsilon_b\rangle\langle\epsilon_b|) = (U_a^\dagger U_b) \otimes (|\epsilon_a\rangle(0)\langle\epsilon_b| = 0$.

Hence, we have $U^\dagger U = (\sum_a U_a^\dagger \otimes (|\epsilon_a\rangle\langle\epsilon_a|)^\dagger)(\sum_b U \otimes (|\epsilon_b\rangle\langle\epsilon_b|)) = \sum_a I \otimes |\epsilon_a\rangle\langle\epsilon_a| = I \otimes I = I$.

9.2 As a bit phase flip channel Y is composed of a Z gate and an X gate, we have circuit:



Let $\rho = \frac{1}{2}(I - c_x\sigma_x + c_y\sigma_y - c_z\sigma_z)$. Then

$$\mathcal{E}(\rho) = (1-p)\rho + pY\rho Y^\dagger = \frac{1}{2}\begin{bmatrix} 1+(1-2p)c_z & (1-2p)c_x - c_y i \\ (1-2p)c_x - c_y i & 1-(1-2p)c_z \end{bmatrix}.$$

The quantum operation has produced a mixture of the Bloch vector states (c_x, c_y, c_z) and $(-c_x, c_y, -c_z)$ with weights $1-2p$ and p respectively. We see that complex part of the matrix is not affected by p. With the phase relaxation process, radius of the Bloch sphere is reduced along the $x-$axis and the $z-$axis to $|1-2p|$, resulting in the shape we see in Figure 9.2(d).