Math 410/510 Introduction to Quantum Computing Homework 2

Sample solution (based on that of Ren He)

1.8 $(cA)^\dagger = (\overline{cA})^t = \overline{c}(\overline{A})^t = (c*)(A*)^t = c^* A^\dagger$;

$(A+B)^\dagger = (\overline{A+B})^t = (\overline{A})^t + (\overline{B})^t = A^\dagger + B^\dagger$;

$(AB)^\dagger = (\overline{AB})^t = \overline{B}^t \overline{A}^t = B^\dagger A^\dagger$.

1.9 Let $\det(\sigma_y - \lambda I) = \lambda^2 - (\frac{1}{\sqrt{2}})^2(1-i^2) = 0, \lambda = \pm 1$. Solving $\frac{1}{\sqrt{2}} \begin{bmatrix} 0, 1+i \\ 1-i, 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x \\ y \end{bmatrix}$, we see that the eigenvectors are $\frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\pi/4} \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\pi/4} \\ -1 \end{bmatrix}$ (normalized).

Then we can let $U = (|\lambda_1\rangle, |\lambda_2\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\pi/4} & e^{i\pi/4} \\ 1 & -1 \end{bmatrix}$

1.10 (a) Method 1. If A is skew-Hermitian, then $(iA)^\dagger = -iA^\dagger = iA$ is Hermitian. So, $iA = iUDU^\dagger$ for a real diagonal matrix, and the eigenvalues of iA are those of iD. So, iA has imaginary eigenvalues.

Method 2. Let A be a skew-Hermitian matrix. We have $A^\dagger = -A$. Let $A|\lambda\rangle = \lambda|\lambda\rangle$.

The Hermitian conjugate of this equation is $\langle\lambda|(-A) = -\langle\lambda|A = \lambda^*\langle\lambda|$.

From these equations we have $\langle\lambda|A|\lambda\rangle = -\langle\lambda|(-A)|\lambda\rangle = -\lambda\langle\lambda|\lambda\rangle = \lambda^*\langle\lambda|\lambda\rangle$;

That is, $-\lambda = \lambda^*$. Let $\lambda = a + bi, a, b \in \mathbb{R}$, then $\lambda^* = a - bi$.

$-(a+bi) = a - bi, -a = a, a = 0$.

Therefore, all eigenvalues of a skew-Hermitian matrix can only be pure imaginary.

1.11 Switch columns of U from $|e_1\rangle, |e_2\rangle, |e_3\rangle$ to $|e_3\rangle, |e_1\rangle, |e_2\rangle$, notate new matrix as U'.

U' is block diagonal with diagonal blocks $[i]$ and $i\sigma_x$

Then the original matrix U has eigenvalue $i, i, -i$ and the following (normalized) eigenvectors,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

1.12 Because H is Hermian, $H^\dagger = H$ and $(I \pm iH)^\dagger = I \mp iH$. Note also that $(A^{-1})^\dagger = (A^\dagger)^{-1}$ for any invertible A. Thus,

$$
\begin{aligned}
&U^\dagger U = [(I+iH)(I-iH)^{-1}]^\dagger (I+iH)(I-iH)^{-1} = ((I-iH)^\dagger)^{-1}(I+iH)^\dagger (I+iH)(I-iH)^{-1} \\
=\ &(I+iH)^{-1}(I-iH)(I+iH)(I-iH)^{-1} = (I+iH)^{-1}(I+H^2)(I-iH)^{-1} \\
=\ &(I+iH)^{-1}(I+iH)(I-iH)(I-iH)^{-1} = I.
\end{aligned}
$$

Alternatively, we can write $H = VDV^\dagger$, where $D = \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix}$ is a real diagonal matrix and V is unitary. Then we have $I \pm iH = I \pm iVDV^\dagger = V(I \pm iD)V^\dagger$ so that

$$U = (I+iH)(I-iH)^{-1} = V(I+iD)(I-ID)^{-1}V^\dagger = VFV^\dagger,$$

where $F = \begin{bmatrix} (1+ih_1)/(1-ih_1) & & \\ & \ddots & \\ & & (1+ih_n)/(1-ih_n) \end{bmatrix}$ is a diagonal matrix with diagonal entries $(1+ih_j)/(1-ih_j)$ having moduli one. So, U is unitary.

1.13 $A = \Sigma_i \lambda_i |\epsilon_i\rangle\langle\epsilon_i| = -\begin{bmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{bmatrix} + 3\begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -2i \\ 2i & 1 \end{bmatrix}$

1.16 (a). By observing, we see $A = 2I + \sigma_x$, then we see that eigenvalues of A, $\lambda = 2 \pm 1, \lambda_1 = 3, \lambda_2 = 1$.

Eigenvectors of A are identical to those of σ_x, which are, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b). $A = \Sigma_i \lambda_i |\epsilon_i\rangle\langle\epsilon_i| = 3\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -2i \\ 2i & 1 \end{bmatrix}$.

(c). $\exp(i\alpha A) = \Sigma_i e^{\lambda_i i\alpha} |\epsilon_i\rangle\langle\epsilon_i| = e^{3i\alpha}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + e^{i\alpha}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = e^{2i\alpha}\begin{bmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{bmatrix}$

Xtra If $A = \begin{bmatrix} h_1 & z \\ z^* & h_2 \end{bmatrix}$ is Hermitian, then

$$A = \begin{bmatrix} h_1 & h_3 + ih_4 \\ h_3 - ih_4 & h_2 \end{bmatrix} = \frac{h_1 + h_2}{2} I + \begin{bmatrix} \frac{h_1 - h_2}{2} & h_3 + ih_4 \\ h_3 - ih_4 & \frac{h_2 - h_1}{2} \end{bmatrix} = u_0 I + u_1\sigma_x + u_2\sigma_y + u_3\sigma_z,$$

where $u_0 = \frac{h_1+h_2}{2}, u_1 = h_3, u_2 = h_4, u_3 \frac{h_1 - h_2}{2} \in \mathbb{R}$.

When $u_0 = 1/2$ and $u_1^2 + u_2^2 + u_3^2 = \frac{1}{4}$, $A = \begin{bmatrix} \frac{1}{2} + u_3 & u_1 + u_2 i \\ u_1 - u_2 i & \frac{1}{2} - u_3 \end{bmatrix}$, $\det(A) = u_0^2 - u_3^2 - u_1^2 - u_2^2 = 0$. Because $u_0^2 = \frac{1}{4}$, $u_1^2 + u_2^2 + u_3^2 = \frac{1}{4}$, we have $det(A) = \frac{1}{4} - \frac{1}{4} = 0$. So, A has eigenvalues $1, 0$ and equals $|\lambda_1\rangle\langle\lambda_1|$ for a unit vector $|\lambda_1\rangle$. So A is a rank one orthogonal projection.

Conversely, if A is a rank one orthogonal projection, then $A = |\lambda_1\rangle\langle\lambda_1|$. Then $1 = \mathrm{tr}(A) = 2u_0$ and $0 = \det(A) = (u_0^2 - u_3^2) - u_1^2 - u_2^2$. So, $u_0 = 1/2$ and $u_1^2 + u_2^2 + u_3^2 = u_0^2 = 1/4$.