Math 410 Intro to Quantum Computing Homework 3

Sample solution based on that of Ren He

1.15 (a) Using code `"A = [5,-2,-4;-2,2,2;-4,2,5]"`, then `"eig(A)"` in matlab, we can find that the eigenvalues of matrix A are $1,1,10$. In fact, using $[U,D] = \text{eig}(A)$, we can get unitary D and diagonal D such that $A = UAU^\dagger$. By direct computation or using U, we get the corresponding orthonormal eigenvectors $|v_1\rangle = \frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}, |v_2\rangle = \frac{1}{3}\begin{bmatrix}-1\\2\\-2\end{bmatrix}, |v_3\rangle = \frac{1}{3}\begin{bmatrix}-2\\1\\2\end{bmatrix}$.

(b) $A = 1P_1 + 10P_2$, where $P_1 = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$ $P_2 = |v_3\rangle\langle v_3|$.

(c) $A^{-1} = \frac{1}{1}P_1 + \frac{1}{10}P_2 = \begin{bmatrix}5 & 2 & 4\\2 & 8 & -2\\4 & -2 & 5\end{bmatrix} + \frac{1}{10}\begin{bmatrix}4 & -2 & -4\\-2 & 1 & 2\\-4 & 2 & 4\end{bmatrix} = \frac{1}{10}\begin{bmatrix}6 & 2 & 4\\2 & 9 & -2\\4 & -2 & 6\end{bmatrix}$.

1.16 Let $A = n\cdot\sigma = \begin{bmatrix}n_z & n_x - in_y\\n_x + in_y & -n_z\end{bmatrix}$. Eigenvalues of A are $\lambda_1 = 1, \lambda_2 = -1$.

By spectral decomposition as done in prop.2, we have

$$P_1 = \frac{1}{2}\begin{bmatrix}1+n_z & n_x - in_y\\n_x + in_y & 1-n_z\end{bmatrix},$$

where $P_2 = \frac{1}{2}\begin{bmatrix}1-n_z & -n_x + in_y\\-n_x - in_y & 1+n_z\end{bmatrix}$, $n\cdot\sigma = A = (1)P_1 + (-1)P_2$. So,

$$\begin{aligned} f(\alpha n\sigma) &= f(\alpha A) = f(\alpha(P_1 - P_2)) = f(\alpha)P_1 - f(\alpha)P_2 \\ &= f(\alpha)\frac{1}{2}(I+A) + f(-\alpha)\frac{1}{2}(I-A) = \frac{f(\alpha)+f(-\alpha)}{2}I + \frac{f(\alpha)+f(-\alpha)}{2}n\cdot\sigma. \end{aligned}$$

1.17 Note that $A^\dagger A = \begin{bmatrix}1 & -i\\0 & 0\\-i & 1\end{bmatrix}\begin{bmatrix}1 & 0 & i\\i & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\0 & 0 & 0\\0 & 0 & 2\end{bmatrix}$, solving with the `"eig()"` function in matlab, we see that the eigenvalues of $A^\dagger A$ are $0,2,2$. So, $\Sigma = \begin{bmatrix}\sqrt{2} & 0 & 0\\0 & \sqrt{2} & 0\end{bmatrix}$. To find the eigenvectors, we solve $\begin{bmatrix}2 & 0 & 0\\0 & 0 & 0\\0 & 0 & 2\end{bmatrix}x = 0$, to get $|\lambda_1\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$; we solve $\begin{bmatrix}0 & 0 & 0\\0 & 2 & 0\\0 & 0 & 0\end{bmatrix}x = 0$ to get $|\lambda_2\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, |\lambda_3\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$. So $V = V^\dagger = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$. Now, $|u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & i\\i & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}, |u_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & i\\i & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}i\\1\end{bmatrix}$.

Hence $U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\i & 1\end{bmatrix}$ and

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\i & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0 & 0\\0 & \sqrt{2} & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}.$$

1.22 We show that $|u_j u_k\rangle$ is an eigenvector:

$(A \otimes I_p + I_m \otimes B)|u_j v_k\rangle = (A|u_j\rangle) \otimes (I_p|v_k\rangle) + (I_m|u_j\rangle) \otimes (B|v_k\rangle)$

$= (\lambda_j|u_j\rangle) \otimes |v_k\rangle + |u_j\rangle \otimes (\mu_k|v_k\rangle) = \lambda_j(|u_j v_k\rangle) + \mu_k(|u_j v_k\rangle) = (\lambda_j + \mu_k)(|u_j v_k\rangle)$.

Therefore, $(A \otimes I_p + I_m \otimes B)$ has the eigenvalues $\{\lambda_j + \mu_k\}$ with the corresponding eigenvectors $\{|u_j v_k\rangle\}$

2.1 (a) Let $\langle\psi|AB|\psi\rangle = z_1$. Because A, B are Hermitian, $z_1^* = \langle\psi|AB|\psi\rangle^\dagger = \langle\psi|B^\dagger A^\dagger|\psi\rangle = \langle\psi|BA|\psi\rangle = z_2$ so that $|z_1| = |z_2|$. Consequently,

$$\begin{aligned} & |\langle\psi|AB - BA|\psi\rangle|^2 + |\langle\psi|AB + BA|\psi\rangle|^2 \\ = \; & |\langle\psi|AB|\psi\rangle - \langle\psi|BA|\psi\rangle|^2 + |\langle\psi|AB|\psi\rangle + \langle\psi|BA|\psi\rangle|^2 \\ = \; & |z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2 = 4|z_1|^2 = 4|\langle\psi|AB|\psi\rangle|^2. \end{aligned}$$

(b) Let $tA + e^{i\theta B}$, where θ satisfies $|\langle\psi|e^{i\theta}AB|\psi\rangle| = e^{i\theta}\langle\psi|AB|\psi\rangle = e^{-i\theta}\langle\psi|BA|\psi\rangle = |\langle\psi|BA|\psi\rangle|$. Then

$$0 \le \langle\psi|(tA + e^{i\theta}B)^\dagger(tA + e^{-i\theta}B)|\psi\rangle = at^2 + 2bt + c$$

where

$$a = \langle\psi|A^2|\psi\rangle, \; b = |\langle\psi|AB|\psi\rangle| = |\langle\psi|BA|\psi\rangle|, \; c = \langle\psi|B^2|\psi\rangle.$$

Since $t^2a + 2tb + c \ge 0$ for all $t \in \mathbb{R}$, $ab - 4b^2 \le 0$, i.e., $\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle \ge |\langle\psi|AB|\psi\rangle|^2$.

Alternatively, let $A|\psi\rangle = |u\rangle, B|\psi\rangle = |v\rangle$. Then $|u\rangle = \alpha|v\rangle + |w\rangle$, where $\alpha = \langle u|v\rangle/\langle v|v\rangle$, and

$$\langle u|u\rangle = |\alpha|^2\langle v|v\rangle + \langle w|w\rangle \ge |\alpha|^2\langle v|v\rangle = |\langle u|v\rangle|^2/\langle v|v\rangle.$$

Thus, $\langle u|u\rangle\langle v|v\rangle \ge |\langle u|v\rangle|^2$, i.e., $\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle \ge |\langle\psi|AB|\psi\rangle|^2$.

(c) By part (a) and (b), we have

$$\langle\psi|[A, B]|\psi\rangle|^2 \le 4|\langle\psi|AB|\psi\rangle|^2 \le |\langle\psi|A^2|\psi\rangle| + |\langle\psi|B^2|\psi\rangle|.$$

(d) Let $\hat{A} = A - \langle\psi|A|\psi\rangle, \hat{B} = B - \langle\psi|B|\psi\rangle$. Then

$$[\hat{A}, \hat{B}] = (A - \langle A\rangle I)(B - \langle B\rangle I) - (B - \langle B\rangle I)(A - \langle A\rangle I) = AB - BA = [A, B].$$

Also,

$$\langle\psi|\hat{A}^2|\psi\rangle = \langle(A - \langle A\rangle I)^2\rangle = \langle A^2 - 2\langle A\rangle A + \langle A\rangle^2 I\rangle = \langle A^2\rangle - \langle A\rangle^2 = (\Delta(A))^2.$$

Similarly, $\Delta B = \sqrt{\langle\psi|\hat{B}^2|\psi\rangle}$. Using part(c), we have

$$\Delta(A)\Delta(B) = \sqrt{\langle\psi|\hat{A}^2|\psi\rangle}\sqrt{\langle\psi|\hat{B}^2|\psi\rangle} \ge \sqrt{\frac{1}{4}(\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle)^2} = \frac{1}{2}\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle = \frac{1}{2}\langle\psi|[A, B]|\psi\rangle.$$

Xtra By theorem, any matrix can be written in form of $USU^\dagger$, where U is unitary and S is upper triangular. Let $A = USU^\dagger, B = VD_2V^\dagger$ for some unitary U, V and triangular S, T. So $A \otimes I_p + I_m \otimes B = USU^\dagger \otimes I_p + I_m \otimes VTV^\dagger = (US \otimes VI_p + UI_m \otimes VT)(U \otimes V)^\dagger = (U \otimes V)(S \otimes I_p + I_m \otimes T)(U \otimes V)^\dagger$

Therefore, $A \otimes I_p + I_m \otimes B = (U \otimes V)(S \otimes I_p + I_m \otimes T)(U \otimes V)^\dagger$