Math 410 Intro to Quantum Computing Homework 4

Sample solution based on that of Ren He

2.2 (*a*) $|\psi(t)\rangle = \exp(i\frac{\omega}{2}\sigma_y t)|\psi(0)\rangle = \begin{bmatrix} \cos\frac{\omega t}{2} & \sin\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\frac{\omega t}{2} \\ \cos\frac{\omega t}{2} \end{bmatrix}$.

(*b*) σ_z has eigenvector $[1,0]^t$ for eigenvalue 1; $p(t) = |\langle z_{\sigma=+1}|\psi(t)\rangle|^2 = |\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \sin\frac{\omega t}{2} \\ \cos\frac{\omega t}{2} \end{bmatrix}|^2 = \sin^2\frac{\omega t}{2}$.

(*c*) σ_x has eigenvector $[1,1]^t/\sqrt{2}$ for eigenvalue 1;

$$p(t) = |\langle x_{\sigma=+1}|\psi(t)\rangle|^2 = |\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sin\frac{\omega t}{2} \\ \cos\frac{\omega t}{2} \end{bmatrix}|^2 = \frac{1}{2}(\cos\frac{\omega t}{2} + \sin\frac{\omega t}{2})^2 .$$

2.3 Consider a positive-semidefinite matrix A, such that for all $|\psi\rangle \in \mathcal{H}, \langle\psi|A|\psi\rangle \geq 0$.

Consider $|\lambda_i\rangle$ an eigenvector of A, then we have $\langle\lambda_i|A|\lambda_i\rangle = \langle\lambda_i|\lambda_i|\lambda_i\rangle = \lambda_i\langle\lambda_i|\lambda_i\rangle$.

We have $\langle\lambda_i|\lambda_i\rangle \geq 0$. Because $\langle\lambda_i|A|\lambda_i\rangle = \langle\lambda_i|\lambda_i|\lambda_i\rangle \geq 0, \lambda_i \geq 0$.

Therefore, every eigenvalue λ_i of A must be non-negative.

Conversely, consider $\langle\psi|A|\psi\rangle = \langle\psi|UDU^\dagger|\psi\rangle$, where U is unitary, and D is diagonal.

Then we have $\langle\psi|A|\psi\rangle = \Sigma_{i=1}^n \lambda_i|c_i|^2 \geq 0$, where $U^\dagger|\psi\rangle = (c_1, \ldots, c_n)^t$.

2.4 Given that ρ is pure, we have $\rho = |\psi\rangle\langle\psi|$ and that $\rho^2 = \rho$, so $\mathrm{tr}(\rho^2) = \mathrm{tr}(\rho) = 1$.

Conversely, given that $\mathrm{tr}(\rho^2) = 1$, as ρ is Hermitian, $\rho = \Sigma_i\lambda_i|\lambda_i\rangle\langle\lambda_i|$, and $\rho^2 = \Sigma_i\lambda_i^2|\lambda_i\rangle\langle\lambda_i|, \lambda_i \geq 0$. Since $0 \leq \lambda_i \leq 1$, we have that $\lambda_i - \lambda_i^2 \geq 0$ for all i. So, $0 = \mathrm{tr}(\rho) - \mathrm{tr}(\rho^2) = \sum_i(\lambda_i - \lambda_i^2)$, and hence $\lambda_i - \lambda_i^2 = 0$, i.e., $\lambda_i = 1$ or $\lambda_i = 0$, for all i. Since $\sum_i \lambda_i = 1$, we see that only one λ_i's equals 1. Hence, $\rho = |\lambda_i\rangle\langle\lambda_i|$ is pure.

2.5 Find partial transpose of ρ_1, we have $\rho_1^{pt} = \begin{bmatrix} \frac{1+p}{4} & 0 & 0 & 0 \\ 0 & \frac{1-p}{4} & \frac{p}{2} & 0 \\ 0 & \frac{p}{2} & \frac{1-p}{4} & 0 \\ 0 & 0 & 0 & \frac{1+p}{4} \end{bmatrix}$;

Solving with matlab, we first built a arbitrary variable p with `"syms p"`, then input ρ_1^{pt} as matrix A.

Then solve for eigenvalues and vectors with `"[v, lambda] = eig(A)"`, we obtain eigenvalues $\frac{1+p}{4}, \frac{1+p}{4}, \frac{1+p}{4}, \frac{1-3p}{4}$.

If $p \leq 1/3$ then $(1-3p)/4 \geq 0$ and hence $(\sum_{j=1}^4 |\lambda_j| - 1)/2 = (\sum_{j=1}^4 \lambda_j - 1)/2 = 0$.

If $p > \frac{1}{3}$ then $\sum(|\lambda_j| - 1)/2 = [\frac{1}{4}(3(1+p) + (3p-1)) - 1]/2 = (3p-1)/4 > 0$.

2.6 Find partial transpose of ρ_1, we have $\rho_2^{pt} = \begin{bmatrix} \frac{p}{2} & 0 & 0 & \frac{1-p}{2} \\ 0 & \frac{1-p}{2} & \frac{p}{2} & 0 \\ 0 & \frac{p}{2} & \frac{1-p}{2} & 0 \\ \frac{1-p}{2} & 0 & 0 & \frac{p}{2} \end{bmatrix}$;

Solving with matlab, we first built a arbitrary variable p with `"syms p"`, then input ρ_2^{pt} as matrix B.

Then solve for eigenvalues and vectors with `"[v, lambda] = eig(B)"`, we obtain eigenvalues $\frac{1}{2}, \frac{1}{2}, \frac{1-2p}{2}, \frac{2p-1}{2}$.

If $p = 0$, then all eigenvalues are nonnegative so that $(\sum_{j=1}^4 |\lambda_j| - 1)/2 = (\sum_{j=1}^4 \lambda_j - 1)/2 = 0$. Otherwise, $1-2p < 0$ or $1-2p > 0$ so that $(\sum_{j=1}^4 |\lambda_j| - 1)/2 = 1$.