Math 410 Intro to Quantum Computing Homework 6

Sample Solution

4.1 If $U_{CNOT} = U \otimes V = (u_{ij}V)$ with $U = (u_{ij}), V \in M_2$, then the $(1,1)$ and $(2,2)$ blocks of U_{CNOT} are multiple of each others, which is not the case. So, $U_{CNOT} \neq (U \otimes V)$.

Alternatively, if $U_{CNOT} = U \otimes V$, then $U_{CNOT}|\psi\rangle|\phi\rangle = U_1\psi\rangle U_2\rangle$. But Problem (4.2) show that it is not the case if $\psi\rangle\psi\rangle = (a|0\rangle + b|1\rangle)|0\rangle$ with $a = b = 1/\sqrt{2}$

4.2 Note that $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}a\\0\\b\\0\end{bmatrix} = \begin{bmatrix}a\\0\\0\\b\end{bmatrix} = a|00\rangle + b|11\rangle$.

4.3 (1) $I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1| = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$;

(2) $\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}$.

(3) $(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$.

If $|\psi_i\rangle = \begin{bmatrix}\psi_{i1}\\\psi_{i2}\end{bmatrix}$ for $i = 1, 2$, then $U_{SWAP}|\psi_1\rangle|\psi_2\rangle = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\psi_{11}\psi_{21}\\\psi_{11}\psi_{22}\\\psi_{12}\psi_{21}\\\psi_{12}\psi_{22}\end{bmatrix} = \begin{bmatrix}\psi_{11}\psi_{21}\\\psi_{12}\psi_{21}\\\psi_{11}\psi_{22}\\\psi_{12}\psi_{22}\end{bmatrix} = |\psi_2\rangle|\psi_1\rangle$.

4.4 $W_1 = U_H, U_H U_H^\dagger = I, U_H = W_1$ is unitary.

For W_n where $n > 1$, we have $W_n^\dagger W_n = (U_H^\dagger \otimes U_H^\dagger \otimes \cdots \otimes U_H^\dagger)(U_H \otimes U_H \otimes \cdots \otimes U_H) = U_H^\dagger U \otimes U_H^\dagger U \otimes \ldots \otimes U_H^\dagger U = I_2 \otimes I_2 \otimes \ldots \otimes I_2 = I_{2n}$. Therefore, W_n is unitary.

4.5 Note that $(U_H \otimes U_H)(I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|)(U_H \otimes U_H) =$

$$\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} = U_{CNOT}$$

4.6 $|00\rangle$: $U_{CNOT}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle) = [1\ 0\ 0\ 1]^T$ $|01\rangle$: $U_{CNOT}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle) = [0\ 1\ 1\ 0]^T$

$|10\rangle$: $U_{CNOT}(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle) = [1\ 0\ 0\ -1]^T$ $|01\rangle$: $U_{CNOT}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle) = [0\ 1\ -1\ 0]^T$

Building up a matrix for the circuit, we have $M = (U_H \otimes I)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) =$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&0&-1\\1&0&-1&0\end{bmatrix}$$

4.7 Note that $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}=U_{SWAP};$

As shown in 4.3(c), we have $U_{SWAP}|\psi_1\rangle|\psi_2\rangle = |\psi_2\rangle|\psi_1\rangle$.

4.8 We have $U_{OR} = \begin{bmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\end{bmatrix}$.

$U_{OR}|00\rangle|0\rangle = [0\ 0\ 0\ 0\ 0\ 0\ 1\ 0]^T = |110\rangle$; $U_{OR}|01\rangle|0\rangle = [0\ 0\ 0\ 0\ 0\ 1\ 0\ 0]^T = |101\rangle$;

$U_{OR}|10\rangle|0\rangle = [0\ 0\ 0\ 1\ 0\ 0\ 0\ 0]^T = |011\rangle$; $U_{OR}|11\rangle|0\rangle = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]^T = |001\rangle$;

Therefore, $U_{OR}|x, y, 0\rangle = |\neg x, \neg y, x \vee y\rangle$.

4.9 Note that $U_{CCNOT}|x, y, 0\rangle = |x, y, x \wedge y\rangle$. Thus, $U_{CCNOT}|x, y, 1\rangle = |x, y, \neg(x \wedge y)\rangle$.

4.10 Note that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$. So, the bra vector of $|x\rangle|y\rangle$ is $\langle x|\langle y|$ and $\langle x|\langle y|u\rangle|v\rangle = \langle x|u\rangle\langle y|v\rangle$.

(a) Using the given forms, we have $\langle\Psi|\Phi\rangle = \langle\psi|\langle 0|U^\dagger U|\phi\rangle|0\rangle = \langle\psi|\langle 0|I|\phi\rangle|0\rangle = \langle\psi|\phi\rangle \otimes 1 = \langle\psi|\phi\rangle$,

$\langle\Psi|\Phi\rangle = \langle\psi|\langle 0|U^\dagger|\phi\rangle|\phi\rangle$, $\langle\Psi|\Phi\rangle = \langle\psi|\langle\psi|U|\phi\rangle|\phi\rangle$, $\langle\Psi|\Phi\rangle = (\langle\psi|\langle\psi|)(|\phi\rangle|\phi\rangle) = \langle\psi|\phi\rangle \otimes \langle\psi|\phi\rangle = (\langle\psi|\phi\rangle)^2$.

(b) As we have $\langle\psi|\phi\rangle = (\langle\psi|\phi\rangle)^2$, then either $\langle\psi|\phi\rangle = 0$ or $\langle\psi|\phi\rangle = 1$. That is, $|\psi\rangle$ and $|\phi\rangle$ are either identical or orthogonal. So, $|\psi\rangle, |\phi\rangle$ are chosen satisfy $\langle\psi|\phi\rangle = 1/2$, then the equality fails. So, the proposed U does not exist.