Math 410 Intro to Quantum Computing Homework 8

Sample Solution

6.1 Let $N = 2^n$, $\tilde{f} = (\tilde{f}(0), \ldots, \tilde{f}(N-1))^t$ and $f = (f(0), \ldots, f(N-1))^t$. Then

$$\Sigma_y |\tilde{f}(y)|^2 = \langle \tilde{f}|\tilde{f}\rangle = \langle \tilde{f}(y)|K^\dagger(x,y)K(x,y)|\tilde{f}(y)\rangle = \langle f|f\rangle = \Sigma_x |f(x)|^2 \text{ as } K \text{ is unitary.}$$

6.2 (1) To get $\langle x|x\rangle = 1$, note that$\langle\psi|\psi\rangle = \mathcal{N}^2 \Sigma_x \cos^2 \frac{2\pi x}{N}$. Thus, $\Sigma_x cos^2 \frac{2\pi x}{N} = \Sigma_x \frac{\cos(\frac{4\pi x}{N})+1}{2} = \frac{N}{2}$, and hence, $\mathcal{N} = 2^{-\frac{n-1}{2}}$

(2) The coefficient of $|x\rangle$ in $U_{QFT_n}|\psi\rangle$ equals $\frac{1}{\sqrt{N}\sqrt{N/2}} \Sigma_y e^{-2\pi ixy/N} \cos \frac{2\pi y}{N}$

$= \frac{1}{\sqrt{2}N} e^{-2\pi ixy/N}(e^{2\pi iy/N} + e^{-2\pi iy/N}) = \frac{1}{\sqrt{2}N}(e^{-2\pi i(x+1)y/N} + e^{-2\pi i(x-1)y/N})$,

which is nonzero if and only if $x = 1$ or $N - 1$. So, the expression reduces to $\frac{1}{\sqrt{2}N} N(\delta_{x,1} + \delta_{x,N-1}) = \frac{1}{\sqrt{2}}(\delta_{x,1} + \delta_{x,N-1})$. Therefore, $U_{QFT_n}|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |N-1\rangle)$.

6.3 Let $\frac{2^n}{P} = m \in \mathbb{N}$. Apply QFT, we obtain $|\Psi'\rangle = \frac{1}{2^n} \Sigma_{x,y} e^{-2\pi ixy/2^n} |y\rangle|f(x)\rangle$.

Separating the summation over x using a arbitrary function $h(x)$, we let

$$\Sigma_{x=0}^{2^n-1} h(x) = \Sigma_{l=0}^{P-1} \Sigma_{k=0}^{m-1} h(kP + l).$$

Substituting, we have $|\Psi'\rangle = \frac{1}{2^n} \Sigma_{l=0}^{P-1} \Sigma_{k=0}^{m-1} \Sigma_y e^{-2\pi ily/2^n} e^{-2\pi iky/m} |y\rangle|f(kP + l)\rangle$

As f has period P, $f(kP + l) = f(l)$, $|\Psi'\rangle = \frac{1}{2^n} \Sigma_y \Sigma_{k=0}^{m-1} e^{-2\pi iky/m} \Sigma_{l=0}^{P-1} e^{-2\pi ily/2^n} |y\rangle|f(l)\rangle$.

When $y = qm, 0 \leq q < P - 1$, then $\Sigma_{k=0}^{m-1} e^{-2\pi iky/m} = \Sigma_{k=0}^{m-1} e^{-2\pi ikq} = m$;

When $y \neq qm$, then $\Sigma_{k=0}^{m-1} e^{-2\pi iky/m} = \frac{1-e^{-2\pi iy}}{1-e^{-2\pi iy/m}} = 0$.

With that, we have $|\Psi'\rangle = \frac{m}{2^n} \Sigma_{l=0}^{P-1} \Sigma_{q=0}^{P-1} e^{-2\pi ilq/P} |qm\rangle|f(l)\rangle$.

Therefore, we can obtain $qm = \frac{q(2^n)}{P}$ for some $q \in \mathbb{Z}, 0 \leq q \leq P - 1$.

7.1 Similar to Proposition 7.1, we have

$U_f|\phi\rangle = DR_f|\phi\rangle = D(\Sigma_{x\neq z}\omega_x|x\rangle - \Sigma_{z\in A}\omega_z|z\rangle) = \Sigma_{x\notin A}(2\bar{\omega} - \omega_x)|x\rangle + \Sigma_{z\in A}(2\bar{\omega} + \omega_z)|z\rangle$.

Alternatively, one can look at the vectors with components $|w\rangle = (w_0, \ldots, w_{N-1})^t$ and see that $|\tilde{w}\rangle = R_f(|w\rangle)$ will change the entries of $|w\rangle$ with labels in A to negative, and then $D|\tilde{w}\rangle = \frac{2}{N}\bar{w}\sum_x |x\rangle - \tilde{w}$. Thus, the entries of $U_f|w\rangle$ has the asserted form.

7.2 By induction on $k \geq 0$. When $k = 0$, the result holds. Assume that the result holds for $k-1$. Then $U_f^k|\varphi_0\rangle = U_f|\varphi_{k-1}\rangle$. By induction assumption $|\varphi_{k-1}\rangle = (w_0, \ldots, w_{N-1})^t$, where $w_x = a_{k-1}$ if $x \in A$ and $w_x = b_{k-1}$ if $x \notin A$. Hence, $|\varphi_k\rangle = U_f(|\varphi_{k-1}\rangle)$ has two types of entries depending on whether the entries have label in A or not:

$$a_k = \frac{2}{N}((N-d)b_{k-1} - da_{k-1}) + a_{k-1} = \frac{1}{N}((N-2d)a_{k-1} + 2(N-d)b_{k-1}),$$

$$b_k = \frac{2}{N}((N-d)b_{k-1} - da_{k-1}) - b_{k-1} = \frac{1}{N}(-2da_{k-1} + (N-2d)b_{k-1}).$$

Remark Note the typo in the book. We should have $a_k = \frac{1}{N}((N-2d)a_{k-1} - 2(N-d)b_{k-1})$.

7.3 Similar to proposition 7.3, we have

Let $p_k = \sqrt{d}a_k, q_k = \sqrt{N-d}b_k$. Then $\begin{bmatrix} p_k \\ q_k \end{bmatrix} = M \begin{bmatrix} p_{k-1} \\ q_{k-1} \end{bmatrix}$, where

$$M = \begin{bmatrix} (N-2d)/N & 2\sqrt{N-d}/N \\ -2\sqrt{N-d}/N & (N-2d)/N \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}.$$

Then $\begin{bmatrix} p_k \\ q_k \end{bmatrix} = M^k \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = \begin{bmatrix} \cos 2k\theta & \sin 2k\theta \\ -\sin 2k\theta & \cos 2k\theta \end{bmatrix} \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \sin[(2k+1)\theta] \\ \cos[(2k+1)\theta] \end{bmatrix}$

As $a_k = \frac{1}{\sqrt{d}}p_k, b_k = \frac{1}{\sqrt{N-d}}q_k$, we have $a_k = \frac{1}{\sqrt{d}}\sin[(2k+1)\theta], b_k = \frac{1}{\sqrt{N-d}}\cos[(2k+1)\theta]$.

7.4 Similar to proposition 7.4, we have

Define $\tilde{m}$ by $(2\tilde{m}+1)\theta = \frac{\pi}{2} \to \tilde{m} = \frac{\pi}{4\theta} - \frac{1}{2}$. We see $|\tilde{m} - m| \leq \frac{1}{2}$ as $m = \lfloor \frac{\pi}{4\theta} \rfloor$.

Then $|(2m+1)\theta - (2\tilde{m}+1)\theta| = |(2m+1)\theta - \frac{\pi}{2} \leq \theta$.

Note that $\theta \sim \frac{d}{\sqrt{N}}$ is small when $N \gg 1$ and $\sin x$ is monotonically increasing in neighborhood of $x = 0$. Thus, $0 < \sin|(2m+1)\theta - \frac{\pi}{2}| < \sin\theta$ or $\cos^2|(2m+1)\theta| \leq \sin^2\theta = \frac{d}{N}$.

Hence, $P_{m,z} = \sin^2[(2m+1)\theta] = 1 - \cos^2[(2m+1)\theta] \geq 1 - \frac{d}{N}$ asserted in (7.47)

Eventually, $m = \lfloor \frac{\pi}{4\theta} \rfloor \leq \frac{\pi}{4\theta} \leq \frac{\pi}{4}\sqrt{\frac{N}{d}}$; so, the operation time of m is $O(\sqrt{\frac{N}{d}})$.