Extreme points of convex matrix sets

Chi-Kwong LI
Department of Mathematics
College of William and Mary

Convex Sets

Definition

A set $\mathcal{S}$ in $\mathbb{R}^N$ is convex if $tx + (1-t)y \in \mathcal{S}$ for any $t \in [0,1], x, y \in \mathcal{S}$.
An element x in a convex set $\mathcal{S}$ is an extreme point if $x \neq (x_1 + x_2)/2$ for any $x_1, x_2 \in \mathcal{S}$.

Krein-Milman Theorem / Caratheodory Theorem

If $\mathcal{S} \subset \mathbb{R}^N$ is a compact (close bounded) convex set, then every element is a convex combination of no more than $N+1$ extreme points.

Examples Polyhedron, circular disk, etc.

Doubly Stochastic Matrices

Birkhoff-von Neumann Theorem

Let $\mathcal{S}_n = \{(a_{ij}) : a_{ij} \geq 0, \sum_i a_{ij} = 1, \sum_j a_{ij} = 1\}$ be the set of doubly stochastic matrices. Then the set of extreme points of $\mathcal{S}$ are the set of permutation matrices.

Proof. One need n^2 equalities from the set of governing equalities and inequalities to determine an extreme points...

Problem

Let $\mathcal{S} = \{(a_{ij}) \in \mathbb{R}^{m \times n} : a_{ij} \geq 0, \sum_{i=1}^{m} a_{ij} = 1, j = 1, \ldots, n\}$ be the set of column stochastic matrices. Show that $A \in \mathcal{S}$ is an extreme points if and only if each column of A has one nonzero entry equal to one.

Open problems

For the following convex sets of matrices, determine the structure of the set of extreme points. In particular, we want to know

* how to check a given element is an extreme point,

* how to construct all the extreme points,

* give a formula for the number (or bounds of the number) of extreme points,

* how to write an elements as a convex combination the smallest number of extreme points.

- the set of $n \times n$ doubly stochastic matrices with each entry bounded by b for some $b \in [1/n, 1]$.
- the set of 3-dimensional doubly stochastic matrices:

$$\mathcal{S} = \{(a_{ijk}) : 1 \leq i, j, k \leq n, a_{ijk} \geq 0, \sum_i a_{ijk} = \sum_j a_{ijk} = \sum_k a_{ijk} = 1\}.$$

- the set of $x = (x_1, \ldots, x_m)^t$ satisfying $Ax \leq b$ for a given $b = (b_1, \ldots, b_n)^t \in \mathbb{R}^n$ and $A = (a_i^{j-1})_{1 \leq i \leq n, 1 \leq j \leq m}$ with $a_1, \ldots, a_n \in \mathbb{R}^n$.