Potential Stability of Matrix Sign Patterns

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The Problem

An $n \times n$ matrix A containing real entries is stable if and only if the real part of each eigenvalue of A is negative.

We consider the following questions:

- What properties of an irreducible $n \times n$ sign pattern matrix allow a stable realization.
- For a given value of n, what is the minimum number of nonzero entries in an irreducible $n \times n$ sign pattern such that it can be potentially stable.

Background Information

Definition: A $k \times k$ **principle minor** of an $n \times n$ matrix A is the determinant of a submatrix of A which is formed by taking the intersection of the $i_1, i_2, \ldots, i_k$-th columns and rows of A. We denote the sum of the $k \times k$ minors of A by $E_k(A)$.

Definition: The **digraph** of an $n \times n$ matrix A is a graph containing n vertices, labeled $1, 2, \ldots, n$, and for every entry of A such that $a_{ij} \neq 0$ there is a directed connection from the i^{th} vertex to the j^{th} vertex.

Routh-Hurwitz Matrix

Let A be an $n \times n$ matrix, with the characteristic polynomial

$$P(t) = t^n + c_1 t^{n-1} + \ldots + c_{n-1} t + c_n$$

Definition: The **Routh-Hurwitz Matrix** of A is defined as the $n \times n$ matrix

$$\begin{bmatrix} c_1 & c_3 & c_5 & \cdots & \cdots \\ 1 & c_2 & c_4 & \cdots & \cdots \\ & c_1 & c_3 & c_5 & \cdots \\ & 1 & c_2 & c_4 & \cdots \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

The matrix A is stable if and only if each of the leading principle minors of its Routh-Hurwitz matrix is positives.

1. The Problem
2. Small Cases
3. Current Research

n=3

Consider the sign pattern below:

$$\begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix}$$

Then for some entries $a_{ij} > 0$, the matrix below is a realization of this sign pattern:

$$A = \begin{bmatrix} -a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & -a_{32} & 0 \end{bmatrix}$$

Then the characteristic polynomial of A is
$P(t) = t^3 + (a_{11})t^2 + (a_{23}a_{32})t + (-a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32})$, and the Routh-Hurwitz matrix of A is

$$\begin{bmatrix} c_1 & c_3 & 0 \\ 1 & c_2 & 0 \\ 0 & c_1 & c_3 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32} & 0 \\ 1 & a_{23}a_{32} & 0 \\ 0 & -a_{11} & -a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32} \end{bmatrix}$$

n=3 (cont.)

$$A = \begin{bmatrix} -a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & -a_{32} & 0 \end{bmatrix}$$

So the leading principle minors of the Routh-Hurwitz matrix for A are

$\Delta_1 = a_{11}$

$\Delta_2 = a_{11}a_{23}a_{32} + (a_{12}a_{23}a_{31} - a_{11}a_{23}a_{32})$

$\Delta_3 = (a_{11}a_{23}a_{32} + (a_{12}a_{23}a_{31} - a_{11}a_{23}a_{32}))(-a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32})$

Since each $a_{ij} > 0$, it is possible to assign values such that $\Delta_1, \Delta_2, \Delta_3 > 0$, making A stable.

n=3

Consider the sign pattern below:

$$\begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ - & - & 0 \end{bmatrix}$$

Then for some entries $a_{ij} > 0$, the matrix below is a realization of this sign pattern:

$$A = \begin{bmatrix} -a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix}$$

Then the characteristic polynomial of A is
$P(t) = t^3 + (a_{11})t^2 + (a_{23}a_{32})t + (a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32})$, and the Routh-Hurwitz matrix of A is

$$\begin{bmatrix} c_1 & c_3 & 0 \\ 1 & c_2 & 0 \\ 0 & c_1 & c_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32} & 0 \\ 1 & a_{23}a_{32} & 0 \\ 0 & -a_{11} & a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32} \end{bmatrix}$$

n=3 (cont.)

$$A = \begin{bmatrix} -a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix}$$

So the leading principle minors of the Routh-Hurwitz matrix for A are

$\Delta_1 = a_{11}$

$\Delta_2 = a_{11}a_{23}a_{32} - (a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32})$

$\Delta_3 = -(a_{11}a_{23}a_{32} - (a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32}))(a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32})$

Note that this gives us

$\Delta_2 = a_{11}a_{23}a_{32} - (a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32}) = -a_{12}a_{23}a_{31} < 0$

However, each $a_{ij} > 0$, so it is impossible for us to assign values to these entries such that $\Delta_2 > 0$. Thus this sign pattern can never be stable.

Method

We currently have an upper bound for the minimum number of edges required for a $n \times n$ sign pattern to be potentially stable. This bound is

$$minimum \leq 2n - (\lfloor \frac{n}{3} \rfloor + 1)$$

The bound is proven for $n \leq 6$ to be the true minimum, leaving $n = 7$ as the next unknown case.

In order to exhaust the options for an $n \times n$ sign pattern with a given number of entries, we begin by working with the digraphs of size n.

Girths

We classify different digraphs by their girths, or the length of the longest continuous cycle within the graph.

Below is listed a formula for the minimum number of edges required for a digraph with n vertices and a girth of k to be strongly connected:

$$e_{n,k} = \begin{cases} 2(n-1) & \text{if } k = 2 \\ ka - 1 & \text{if } n = a(k-1) \text{ for some } a \in \mathbb{Z}^+ \\ ka & \text{if } n = a(k-1) + 1 \text{ for some } a \in \mathbb{Z}^+ \\ ka + b & \text{if } n = a(k-1) + b \text{ for some } a \in \mathbb{Z}^+ \text{ and } 1 < b < k-1 \end{cases}$$

Algorithm for Correct Minors

We have also developed an algorithm which generates a digraph with n vertices which is strongly connected and which has correct minors.

We conjecture that this digraph has the minimum number of edges required for an size n digraph to have correct minors.