Math 410 Quantum Computing **Examination 1** Due: noon, 12 October, 2012

1. (a) Show that if $A = (A_{ij}) \in M_n$ is normal, then $\sum_{j=1}^n |A_{ij}|^2 = \sum_{j=1}^n |A_{ji}|^2$ for $i = 1, \ldots, n$.
(b) Construct an example of a non-normal matrix $A = (A_{ij}) \in M_2$ such that

$$\sum_{j=1}^n |A_{ij}|^2 = \sum_{j=1}^n |A_{ji}|^2 \text{ for } i = 1, 2.$$

2. Let

$$\rho = \frac{1}{8}\begin{pmatrix} 3 & 0 & 2i & 0 \\ 0 & 1 & 0 & i \\ -2i & 0 & 3 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix} \in M_2 \otimes M_2.$$

(a) Find the eigenvalues of ρ and show that ρ is a mixed state.
(b) Compute $\mathrm{Tr}_1(\rho)$ and $\mathrm{Tr}_2(\rho)$.

3. (a) Suppose $|\psi\rangle \in \mathbb{C}^2$ is a unit vector and $\rho = |\psi\rangle\langle\psi|$. Show that $\rho|\psi\rangle = |\psi\rangle$.
(b) Let $|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\psi_3\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\rho = \frac{1}{3}\sum_{j=1}^3 |\psi_j\rangle\langle\psi_j|$.
Find the spectral decomposition of ρ and determine $\log(\rho)$ and $\exp(\rho)$.

4. Suppose

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Compute the fidelity $F(\rho_1, \rho_2)$.
(b) Consider the measurement operator

$$M = \frac{1}{10}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 3 & 2+i & 0 \\ 0 & 2-i & 3 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}.$$

Determine the quantum state after applying the measurement M to ρ_2 in (a).

5. Let $\rho = \frac{1}{4}\begin{pmatrix} 3 & i \\ -i & 1 \end{pmatrix}$.
(a) Show that ρ is a mixed state.
(b) Determine two different $|\Psi_1\rangle, |\Psi_2\rangle \in \mathbb{C}^4 \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$, which are not multiples of each others, such that

$$\mathrm{Tr}_2(\rho_1) = \mathrm{Tr}_2(\rho_2) = \rho \quad \text{for } \rho_j = |\psi_j\rangle\langle\psi_j| \quad \text{with } j = 1, 2.$$

6. Let

$$\rho_1 = \frac{1}{2}\begin{pmatrix} \cos^2 t & 0 & 0 & \cos t \sin t \\ 0 & \sin^2 t & \cos t \sin t & 0 \\ 0 & \cos t \sin t & \cos^2 t & 0 \\ \cos t \sin t & 0 & 0 & \sin^2 t \end{pmatrix}$$

and

$$\rho_2 = \frac{1}{2}\begin{pmatrix} \cos^2 t & 0 & 0 & \cos t \sin t \\ 0 & \sin^2 t & \cos t \sin t & 0 \\ 0 & \cos t \sin t & \sin^2 t & 0 \\ \cos t \sin t & 0 & 0 & \cos^2 t \end{pmatrix}.$$

(a) Show that ρ_1 is separable for all $t \in \mathbb{R}$.

(b) Determine $t \in [0, 2\pi]$ such that ρ_2 is separable.

7. Suppose an mn-by-mn density matrix $A \in M_m \otimes M_n$ is in block form $A = (A_{ij})$ with $A_{ij} \in H_n$. Recall that

$$\mathrm{Tr}_1(A) = \sum_{j=1}^{m} (\langle e_j| \otimes I_n) A (|e_j\rangle \otimes I_n)$$

for a fixed orthonormal basis $\{|e_1\rangle, \ldots, |e_m\rangle\}$ for $\mathbb{C}^m$.

(a) Suppose $\{|e_1\rangle, \ldots, |e_m\rangle\}$ is the standard basis for $\mathbb{C}^m$.
Show that $\langle e_1| \otimes I_n = [I_n|0_n|\ldots|0_n]$, $\langle e_2| \otimes I_n = [0_n|I_n|0_n|\ldots|0_n]$, etc.

(b) Show that $\mathrm{Tr}_1(A) = A_{11} + \cdots + A_{mm}$.

(c) Show that $\mathrm{Tr}_1(A)$ is the same for any choice of orthonormal basis $\{|e_1\rangle, \ldots, |e_m\rangle\}$ for $\mathbb{C}^m$.

(d) Let $\tilde{A}$ be obtained from A by changing A_{ij} to 0 whenever $i \neq j$. Show that $\mathrm{Tr}_1(A) = \mathrm{Tr}_1(\tilde{A})$.

8. (a) Explain why there is or there is no unitary matrix U (quantum gate) such that

$$(U|00\rangle, U|01\rangle, U|10\rangle, U|11\rangle) = \left(|00\rangle, |01\rangle, |10\rangle, \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\right).$$

(b) Explain why there is or there is no unitary matrix V (quantum gate) such that

$$(V|000\rangle, V|111\rangle) = (|GHZ\rangle, |W\rangle) = \left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)\right).$$