Math 410 Quantum Computing **Examination 1** Sample Solution

1. (a) Show that if $A = (A_{ij}) \in M_n$ is normal, then $\sum_{j=1}^n |A_{ij}|^2 = \sum_{j=1}^n |A_{ji}|^2$ for $i = 1, \dots, n$.

   Suppose $A = (A_{ij})$ is normal. Then for each $i = 1, \dots, n$, the (i,i) entry of $AA^\dagger$ is $\sum_{j=1}^n |A_{ij}|^2$ and equals the (i,i) entry of $A^\dagger A$, which is $\sum_{j=1}^n |A_{ji}|^2$. The result follows.

   (b) Construct an example of a non-normal matrix $A = (A_{ij}) \in M_2$ such that

   $$\textstyle\sum_{j=1}^n |A_{ij}|^2 = \sum_{j=1}^n |A_{ji}|^2 \text{ for } i = 1, 2.$$

   Consider $A = \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}$, and $A^\dagger = \begin{bmatrix} 1 & 1 \\ -i & -i \end{bmatrix}$, we have $AA^\dagger = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \neq A^\dagger A = \begin{bmatrix} 2 & 2i \\ -2i & 2 \end{bmatrix}$, and the $(1,1)$ and $(2,2)$ entrie of $AA^\dagger$ and $A^\dagger A$ are the same.

2. Let

   $$\rho = \frac{1}{8}\begin{pmatrix} 3 & 0 & 2i & 0 \\ 0 & 1 & 0 & i \\ -2i & 0 & 3 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix} \in M_2 \otimes M_2.$$

   (a) Find the eigenvalues of ρ and show that ρ is a mixed state.

   Interchanging the second and third rows, and also the second and third columns of the matrix, we get $\begin{pmatrix} 3 & 2i \\ -2i & 3 \end{pmatrix} \oplus \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, which has eigenvalues: $5/8, 1/8, 1/4, 0$. So, ρ is not rank one, and is not a pure state.

   (b) Compute $\mathrm{Tr}_1(\rho)$ and $\mathrm{Tr}_2(\rho)$.

   $\mathrm{Tr}_1(A) = \sum_{j=1}^2 (\langle e_j| \otimes I_n)A(|e_j\rangle \otimes I_n) = \frac{1}{4}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, equals the sum of the diagonal blocks, and

   $\mathrm{Tr}_2(\rho) = \Sigma_{k=1}^2 (I_2 \otimes \langle e_k|)\rho(I_2 \otimes |e_k\rangle) = \frac{1}{8}\begin{bmatrix} 4 & 3i \\ -3i & 4 \end{bmatrix}$, has the traces of the blocks of ρ as entries.

3. (a) Suppose $|\psi\rangle \in \mathbb{C}$ is a unit vector and $\rho = |\psi\rangle\langle\psi|$. Show that $\rho|\psi\rangle = |\psi\rangle$.

   Because $|\psi\rangle$ is a unit vector, $\sqrt{\langle\psi|\psi\rangle} = \langle\psi|\psi\rangle = 1$. So, $\rho|\psi\rangle = |\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle(\langle\psi|\psi\rangle) = |\psi\rangle$.

   (b) Let $|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\psi_3\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\rho = \frac{1}{3}\sum_{j=1}^3 |\psi_j\rangle\langle\psi_j|$. Find the spectral decomposition of ρ and determine $\log(\rho)$ and $\exp(\rho)$.

   $\rho = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| = \begin{bmatrix} \frac{1}{2} & -\frac{i}{6} \\ \frac{i}{6} & \frac{1}{2} \end{bmatrix} = \frac{2}{3}P_1 + \frac{1}{3}P_2$, where $P_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $P_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Thus, $\log(\rho) = \log(\frac{2}{3})P_1 + \log(\frac{1}{3})P_2$ and $\exp(\rho) = \exp(\frac{2}{3})P_1 + \exp(\frac{1}{3})P_2$.

4. Suppose

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Compute the fidelity $F(\rho_1, \rho_2)$.

Note that $\rho_3 = \sqrt{\rho_1}\rho_2\sqrt{\rho_1} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $\sqrt{\rho_3} = \sqrt{\rho_1}\rho_2\sqrt{\rho_1} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Thus, $F(\rho_1, \rho_2) = \mathrm{Tr}(\sqrt{\rho_3}) = 1/\sqrt{2}$.

(b) Consider the measurement operator

$$M = \frac{1}{10}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 3 & 2+i & 0 \\ 0 & 2-i & 3 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}.$$

Determine the quantum state after applying the measurement M to ρ_2 in (a).

Because $\rho_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$, we have $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$.

The state right after measurement is expected to be $\frac{M|\psi\rangle}{\|M|\psi\rangle\|}$. Solving with Matlab, we find it equal to $\frac{1}{\sqrt{5^2+1^2+5^2+1^2}}\begin{bmatrix} 0 & 5+i & 5-i & 0 \end{bmatrix}^T = \frac{1}{\sqrt{52}}\begin{bmatrix} 0 & 5+i & 5-i & 0 \end{bmatrix}^T$.

Alternatively, in density form, we have $\frac{1}{\mathrm{Tr} M\rho M^\dagger} M\rho M^\dagger = \frac{1}{52}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 26 & 24+10i & 0 \\ 0 & 24-10i & 26 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

5. Let $\rho = \frac{1}{4}\begin{pmatrix} 3 & i \\ -i & 1 \end{pmatrix}$.

(a) Show that ρ is a mixed state.

Computing with Matlab, we find $\rho^2 = \frac{1}{8}\begin{pmatrix} 5 & 2i \\ -2i & 1 \end{pmatrix} \neq \rho$. Therefore, ρ is a mixed state.

(b) Determine two different $|\Psi_1\rangle, |\Psi_2\rangle \in \mathbb{C}^4 \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$, which are not multiples of each others, such that $\mathrm{Tr}_2(\rho_1) = \mathrm{Tr}_2(\rho_2) = \rho \quad$ for $\rho_j = |\psi_j\rangle\langle\psi_j| \quad$ with $j = 1, 2$.

We have $\Sigma_k(I_2 \otimes \langle e_k|)\rho_1(I_2 \otimes |e_k\rangle) = \frac{1}{4}\begin{bmatrix} 3 & i \\ -i & 1 \end{bmatrix}$.

Spectral decompose ρ, we have $\rho = 0.1464\begin{bmatrix} .3827i \\ -.9239 \end{bmatrix}\begin{bmatrix} .3827i \\ -.9239 \end{bmatrix}^T + 0.8536\begin{bmatrix} .9239i \\ .3827 \end{bmatrix}\begin{bmatrix} .9239i \\ .3827 \end{bmatrix}^T$;

Let $|\Psi_1\rangle = \sqrt{.1464}\begin{bmatrix} .3827i \\ -.9239 \end{bmatrix} \otimes |e_1\rangle + \sqrt{.8536}\begin{bmatrix} .9239i \\ .3827 \end{bmatrix} \otimes |e_2\rangle = \begin{bmatrix} 0.1464i \\ .8536i \\ -.3535 \\ .3536 \end{bmatrix}$;

Let $|\Psi_2\rangle = \sqrt{.1464}\begin{bmatrix} .3827i \\ -.9239 \end{bmatrix} \otimes (\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} |e_1\rangle) + \sqrt{.8536}\begin{bmatrix} .9239i \\ .3827 \end{bmatrix} \otimes (\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} |e_2\rangle) = \begin{bmatrix} 0.1464i \\ -.8536i \\ -.3535 \\ -.3536 \end{bmatrix}$

This method is proven in Exercise 2.9, and we can easily find

$\text{Tr}_2(\rho_1) = \Sigma_k(I_2 \otimes \langle e_k|)\rho_1(I_2 \otimes |e_k\rangle) = \text{Tr}_2(\rho_2) = \Sigma_k(I_2 \otimes \langle e_k|)\rho_2(I_2 \otimes |e_k\rangle) = \frac{1}{4}\begin{bmatrix} 3 & i \\ -i & 1 \end{bmatrix}$.

Let $|\Psi_1\rangle = \begin{bmatrix} 0.1464i & .8536i & -.3535 & .3536 \end{bmatrix}^T$ and $|\Psi_2\rangle = \begin{bmatrix} 0.1464i & -.8536i & -.3535 & -.3536 \end{bmatrix}^T$ we can have $|\Psi_1\rangle \neq |\Psi_2\rangle$, and that $\text{Tr}_2(\rho_1) = \text{Tr}_2(\rho_2) = \rho$ for $\rho_j = |\psi_j\rangle\langle\psi_j|$ with $j = 1, 2$.

6. Let

$$\rho_1 = \frac{1}{2}\begin{pmatrix} \cos^2 t & 0 & 0 & \cos t \sin t \\ 0 & \sin^2 t & \cos t \sin t & 0 \\ 0 & \cos t \sin t & \cos^2 t & 0 \\ \cos t \sin t & 0 & 0 & \sin^2 t \end{pmatrix}$$

and

$$\rho_2 = \frac{1}{2}\begin{pmatrix} \cos^2 t & 0 & 0 & \cos t \sin t \\ 0 & \sin^2 t & \cos t \sin t & 0 \\ 0 & \cos t \sin t & \sin^2 t & 0 \\ \cos t \sin t & 0 & 0 & \cos^2 t \end{pmatrix}.$$

(a) Show that ρ_1 is separable for all $t \in \mathbb{R}$.

The eigenvalues of $\rho_1^{pt} = \rho_1$ are: $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = \lambda_4 = \frac{1}{2}(\sin^2(t) + \cos^2(t)) = \frac{1}{2}$.

Thus negativity vanishes for all $t \in [0, 2\pi]$. Therefore, ρ_1 is separable.

(b) Determine $t \in [0, 2\pi]$ such that ρ_2 is separable.

The eigenvaleus of $2\rho^{pt}$ are: $\lambda_1 = \cos^2(t) + \sin(t)\cos(t), \lambda_2 = \cos^2(t) - \sin(t)\cos(t), \lambda_3 = \sin^2(t) + \sin(t)\cos(t), \lambda_4 = \sin^2(t) - \sin(t)\cos(t)$.

We consider $t \in [0, \pi/2] \cup (\pi/2, \pi] \cup (\pi, 3\pi/2] \cup (3\pi/2, 2\pi]$ and conclude that ρ_2 is separable, i.e., $\lambda_i \geq 0$ for $i = 1, 2, 3, 4$, if and only if $t = 0, t = \frac{\pi}{4}, t = \frac{\pi}{2}, t = \frac{3\pi}{4}, t = \frac{5\pi}{4}, t = \frac{3\pi}{2}, t = \frac{7\pi}{4}, t = 2\pi$.

7. Suppose an mn-by-mn density matrix $A \in M_m \otimes M_n$ is in block form $A = (A_{ij})$ with $A_{ij} \in H_n$. Recall that $\text{Tr}_1(A) = \sum_{j=1}^m (\langle e_j| \otimes I_n)A(|e_j\rangle \otimes I_n)$ for a fixed orthonormal basis $\{|e_1\rangle, \ldots, |e_m\rangle\}$ for $\mathbb{C}^m$.

(a) Suppose $\{|e_1\rangle, \ldots, |e_m\rangle\}$ is the standard basis for $\mathbb{C}^m$. Show that $\langle e_1| \otimes I_n = [I_n|0_n|\ldots|0_n]$, $\langle e_2| \otimes I_n = [0_n|I_n|0_n|\ldots|0_n]$, etc.

By direct computation, we have $\langle e_1| \otimes I_n = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix} \otimes I_n = \begin{bmatrix}I_n & 0_n & \cdots & 0_n\end{bmatrix}$, $\langle e_2| \otimes I_n = \begin{bmatrix}0_n & I_n & 0_n & \cdots & 0_n\end{bmatrix}, \ldots, \langle e_n| \otimes I_n = \begin{bmatrix}0_n & \cdots & 0_n & I_n\end{bmatrix}$.

(b) Show that $\mathrm{Tr}_1(A) = A_{11} + \cdots + A_{mm}$.

From result of part (a), we have

$\mathrm{Tr}_1(A) = \Sigma_i(\langle e_i| \otimes I_n)A(|e_i\rangle \otimes I_n) = A_{11} + \cdots + A_{mm}$.

(c) Show that $\mathrm{Tr}_1(A)$ is the same for any choice of orthonormal basis $\{|e_1\rangle, \ldots, |e_m\rangle\}$ for $\mathbb{C}^m$.

If $A = B \otimes C$, and $\{|u_1\rangle, \ldots, |u_m\rangle\}$ is an orthonormal basis for $\mathbb{C}^m$, then

$$\mathrm{Tr}_1(A) = \Sigma_i(\langle e_i|\otimes I_n)A(|e_i\rangle\otimes I_n) = \Sigma_i(\langle u_i|\otimes I_n)(B\otimes C)(|u_i\rangle\otimes I_n) = \Sigma_i\langle u_i|B|u_i\rangle\otimes C = (\mathrm{Tr}B)C,$$

which is the same for any choice of orthonormal basis $\{|u_1\rangle, \ldots, |u_m\rangle\}$. In general, we can use a basis $\{B_1, \ldots, B_r\}$ for M_m and a basis $\{C_1, \ldots, C_s\}$ for M_n to form a basis $\{B_i \otimes C_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ In general, every $A \in M_{mn}$ is a linear combination of matrices of the form $B_r \otimes C_s$, say, $A = \sum_{r,s} c_{ij} B_i \otimes C_s$. Then $\mathrm{Tr}_1(A) = \sum_{r,s} \mathrm{Tr}_1(B_r \otimes C_s) = \sum_{r,s} c_{ij} \mathrm{Tr}(B_r) C_s$ is independent of the choice of the orthonormal basis $\{|e_1\rangle, \ldots, |e_m\rangle\}$.

(d) Let $\tilde{A}$ be obtained from A by changing A_{ij} to 0 whenever $i \neq j$.

Show that $\mathrm{Tr}_1(A) = \mathrm{Tr}_1(\tilde{A})$.

As proven in part (b), $\mathrm{Tr}_1(A) = A_{11} + \cdots + A_{mm}$, that is, any A_{ij} where $i \neq j$ would not affect value of $\mathrm{Tr}(A)$. Therefore, $\mathrm{Tr}(\tilde{A}) = \mathrm{Tr}(A)$.

8. (a) Explain why there is or there is no unitary matrix U (quantum gate) such that

$$(U|00\rangle, U|01\rangle, U|10\rangle, U|11\rangle) = \left(|00\rangle, |01\rangle, |10\rangle, \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\right).$$

By the given condition, we have $U = U[|00\rangle\ |01\rangle\ |10\rangle\ |11\rangle] = \begin{pmatrix}1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2\end{pmatrix}$, which is clearly not a unitary.

(b) Explain why there is or there is no unitary matrix V (quantum gate) such that

$$(V|000\rangle, V|111\rangle) = (|GHZ\rangle, |W\rangle) = \left(\tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \tfrac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)\right).$$

Note that the set the matrix $[|GHZ\rangle, |W\rangle\rangle, |e_3\rangle, \ldots, |e_8\rangle]$ is in lower triangular form with determinant $1/\sqrt{6}$. So, the column are linear independent. Because $\{|GHZ\rangle, |W\rangle\}$ is an orthonormal pair, we can apply Gram-Schmidt process to the set to get an orthonormal basis $\{|GHZ\rangle, |W\rangle, |v_3\rangle, \ldots, |v_8\rangle\}$. Then $V = [|GHZ\rangle\ |v_3\rangle\ \cdots\ |v_8\rangle\ |W\rangle]$ satisfies the requirement.