This is a generalizations of the use of Hadamard gate to compute $W_n \sum_x |x\rangle$ and $W_n \sum_x |f(x)\rangle$.

6.1 Quantum Integral Transform

Let $S_n = \{0, \ldots, N-1\}$ with $N = 2^n$ and let K be an $N \times N$ complex matrix with entries $K(i,j)$ with $i, j \in S_n$. Then K is a QIT transform converting $f = (f(0), \ldots, f(N-1))^t$ to $\tilde{f} = (\tilde{f}(0), \ldots, \tilde{f}(N-1))^t$ by $\tilde{f} = Kf$.

If K is unitary (invertible) then

$$f = K^\dagger \tilde{f} \qquad \text{(respectively, } f = K^{-1}\tilde{f}\text{)}.$$

Proposition If $U|x\rangle = K|y\rangle$, then

$$U\left[\sum_{x=0}^{2^n-1} f(x)|x\rangle\right] = \sum_{y=0}^{2^n-1} \tilde{f}(y)|y\rangle.$$

6.2 Quantum Fourier Transform

Suppose $N = 2^n$, $w = e^{2\pi i/N}/\sqrt{N}$ and $K = K(x, y)$ with $K(x, y) = (w_n^{-xy})$.

Then $\tilde{f} = Kf$ is a commonly used QFT.

Example When $n = 1, 2$.

6.3 Application of QFT to period finding

This is an essential component in the Shor's algorithm.

For a periodic function, $f : S_n \to S_n$, where $S_n = \mathbb{Z}_2^n$, we want to detect $P \in S_n$ such that

$$f(x) = f(x + P) \quad \text{for all } x \in S_n.$$

Example Let $n = 3, P = 2$; $f(0) = f(2) = f(4) = f(6) = a, f(1) = f(3) = f(5) = f(7) = b$.

Step 1. Prepare $|\Psi_0\rangle = |0\rangle|0\rangle \in S_3 \otimes S_3$.
Step 2. Apply $W_3 \otimes I_8$ to $|\Psi_0\rangle$ and the oracle U_f to get $|\Psi\rangle = \gamma \sum_x |x\rangle|f(x)\rangle$.
Step 3. Apply $F = [e^{-2\pi ixy/8}] \otimes I_n$ to $|\Psi\rangle$ to get

$$\begin{aligned}
|\Psi'\rangle &= \gamma \sum_{x,y} e^{-2\pi ixy/8} |y, f(x)\rangle \\
&= \gamma|0\rangle[|f(0)\rangle + |f(1)\rangle + \cdots + f(7)\rangle] \qquad (y = 0) \\
&\quad +\gamma|1\rangle[|f(0)\rangle + e^{-2\pi i/8}|f(1)\rangle + \cdots + e^{-2\pi i7/8} f(7)\rangle] \qquad (y = 1) \\
&\quad + \qquad \cdots \quad \cdots \\
&\quad +\gamma|1\rangle[|f(0)\rangle + e^{-14\pi i/8}|f(1)\rangle + \cdots + e^{-14\pi i7/8} f(7)\rangle] \qquad (y = 7) \\
&= \frac{1}{2}(|0, a\rangle + |0, b\rangle + |4, a\rangle + e^{-i\pi}|4, b\rangle).
\end{aligned}$$

Step 4. Measurement of the first register gives $0, 4$. So the period is 2.

Remark In general, the observed value of the first register is one of

$$\frac{1}{P} k \cdot 2^n, \qquad k = 0, 1, \ldots, P - 1.$$

6.4 Implementation of QFT

We need the controlled B_{jk} gate corresponds to $U_{jk}|x,y\rangle = e^{-i\theta_{jk}xy}|x,y\rangle$ for $|x,y\rangle \in S_2$, and the Swap gate.

Proposition QFT can be implemented using $O(n^2)$ elementary gates.

6.5 Walsh-Hadamard Transform

The kernel $W_n = ((-1)^{x\cdot y})$ defines the discrete integral transform

$$\tilde{f}(y) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{x\cdot y} f(x).$$

6.6 Selective Phase Rotation Transform

The kernel $\mathrm{diag}\,(\theta_0, \ldots, \theta_{N-1})$ defines the transform

$$\tilde{f}(y) = \sum e^{i\theta_x} \delta_{xy} f(x) = e^{i\theta_y} f(y).$$