§7.1 Search for a single file

Let $f : S_n \to \{0,1\}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = z, \\ 0, & \text{if } x \neq z. \end{cases}$$

Step 1 Define the reflection R_f such that $R_f = I - 2|z\rangle\langle z|$. We have

$$R_f f = \sum_x f(x) R_f |x\rangle = \sum_x (-1)^{f(x)} |x\rangle.$$

Step 2 Construct $D = -I + 2|\varphi_0\rangle\langle\varphi_0|$ with $|\varphi_0\rangle = \sum_{x=0}^{N-1} |x\rangle/\sqrt{N}$.

Step 3 Construct $U_f = DR_f$ and its action on $|\varphi\rangle = \sum_x w_x |x\rangle$ with $\sum_x |w_x|^2 = 1$. Then

$$U_f^k |\varphi_0\rangle = a_k |z\rangle + b_k \sum_{x \neq z} |x\rangle$$

such that $a_0 = b_0 = 1/\sqrt{N}$. For $k \geq 1$ we have

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{N} \begin{pmatrix} N-2 & 2(N-1) \\ -2 & N-2 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}.$$

Here note that $U_f[b_k, \ldots, b_k, a_k, b_k, \ldots, b_k]^t = [b_{k+1}, \ldots, b_{k+1}, a_{k+1}, b_{k+1}, \ldots, b_{k+1}]^t$.

Let $c_k = \sqrt{N-1} b_k$. If $(a_0, c_0) = (1, \sqrt{N-1})/\sqrt{N} = (\sin\theta, \cos\theta)$, then

$$\begin{pmatrix} a_k \\ c_k \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{k-1} \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} \sin[(2k+1)\theta] \\ \cos[(2k+1)\theta] \end{pmatrix}.$$

Step 4 Maximize $P_{z,k}^2 = a_k^2$ by putting $(2k+1)\theta \approx \pi/2$. For large N we have $m = \lfloor \pi/4\theta \rfloor$ so that $m = O(\sqrt{N})$.

§7.2 Search for d files

Let $A \subseteq S_n$ have d elements, and $f : S_n \to \{0,1\}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Step 1 Define the reflection R_f such that

$$R_f = I - 2\sum_{z \in A} |z\rangle\langle z|.$$

Then for $|\varphi\rangle = \sum_{x=0}^{N-1} w_x|x\rangle$, $R_f(\varphi) = \sum_{x \notin A} w_x|x\rangle - \sum_{z \in A} w_z|z \in A\rangle$.

Step 2 Construct $D = -I + 2|\varphi_0\rangle\langle|\varphi_0|$ with $|\varphi_0\rangle = \sum_{x=0}^{N} |x\rangle/\sqrt{N}$.

Step 3 Construct $U_f = DR_f$ and its action on $|\varphi\rangle = \sum_x w_x|x\rangle$ with $\sum_x |w_x|^2 = 1$.

Then $U_f^k|\varphi_0\rangle = a_k \sum_{z \in A} |z\rangle + b_k \sum_{x \notin A} |x\rangle$ such that $a_0 = b_0 = 1/\sqrt{N}$ and for $k \geq 1$ we have

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{N} \begin{pmatrix} N - 2d & 2(N-d) \\ -2d & N - 2d \end{pmatrix} \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}$$

so that

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} (\sin[(2k+1)\theta])/\sqrt{d} \\ (\cos[(2k+1)\theta])/\sqrt{N-d} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sqrt{d/N} \\ \sqrt{1 - d/N} \end{pmatrix}.$$