9.1 Open Quantum System

A unitary time evolution of a close system is determined by the quantum map $\mathcal{E}$ defined by

$$\mathcal{E}(\rho_S) = U(t)\rho_S U(t)^\dagger.$$

Here, ρ_S is the density matrix of a closed system at time $t = 0$ and $U(t)$ is the time evolution operator.

An open system is a system of interest (called the **principal system**) coupled with its environment. The total Hamiltonian is given by

$$H_T = H_S + H_E + E_{SE},$$

where H_S, H_E and H_{SE} are the system Hamiltonian, the environment Hamiltonian and their interaction Hamiltonian, respectively.

The state of the total system, which is assumed to be closed, will be described by ρ acting on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$ such that

$$\rho(0) = \rho_S \otimes \rho_E \quad \text{and} \quad \rho(t) = U(t)(\rho_S \otimes \rho_E)U(t)^\dagger \ \text{ for } t > 0.$$

We study the system $(\mathcal{H}_S)$ by taking the partial trace

$$\rho_S(t) = \mathrm{Tr}_E[U(t)(\rho_S \otimes \rho_E)U(t)^\dagger].$$

We may assume $\rho_E = |\varepsilon_0\rangle\langle\varepsilon_0|$ by purification. Then

$$\rho_S(t) = \sum_a E_a(t)\rho_S E_a(t)^\dagger \quad \text{with} \quad E_a(t) = \langle\varepsilon_a|U(t)|\varepsilon_0\rangle.$$

This is known as the operator-sum representation of the quantum operation $\mathcal{E}$. The operators $E_a(t)$ are known as the **Kraus operators** of the quantum operation $\mathcal{E}$. Note that

$$\sum_a E_a(t)^\dagger E_a(t) = I.$$

It is also possible to define a quantum operation such that $\rho_E \to \rho_S(t)$ such as

$$\rho_S(t) = \mathrm{Tr}_E[U(t)(e_0\rangle\langle e_0| \otimes \rho_E)U(t)^\dagger].$$

Noisy quantum channels are quantum operations.

For example, suppose U_a is unitary, $p_a \in (0,1]$, and $\sum_a p_a = 1$. A **mixing process** is defined by

$$\mathcal{M}(\rho_S) = \sum_a p_a U_a \rho_S U_a^\dagger.$$

Remark The description of quantum operations and noisy quantum channels are interchangeable.

Completely positive linear maps.

Definition A linear map (function) Λ on matrices such that $\Lambda \otimes I_r$ maps positive operators to positive operators is call **completely positive**.

Theorem A map on matrices is completely positive if and only if it admits an operator-sum representation:

$$A \mapsto \sum_j E_j A E_j^\dagger.$$

The matrices E_j are called the Choi/Kraus operators. In the context of quantum error correction, E_j are called the error operators.

9.2 Measurements as quantum operations

Projective measurements as quantum operations.

Let $A = \sum_j \lambda_j P_j$ so that the measurement of A in a state ρ is

$$p(j) = \mathrm{Tr}(P_j \rho P_j) = \mathrm{Tr}(P_j \rho)$$

and state is changed to

$$\rho \to P_j \rho P_j / p(j).$$

Then the measurement process is the quantum operation

$$\rho_S \mapsto \sum_j p(j) \frac{P_j \rho_S P_j}{p(j)} = \sum_j P_j \rho_S P_j.$$

Positive Operator-valued measure (POVM)

Suppose $|\psi\rangle|e_0\rangle$ is the state of an open system. Let U be a unitary operator acting on the system such that

$$|\Psi\rangle = U|\psi\rangle|e_0\rangle = \sum_j M_j|\psi\rangle|e_j\rangle.$$

Then

$$1 = \langle e_0|\langle\psi|U^\dagger U|\psi\rangle|e_0\rangle = \langle\psi|\sum_j M_j^\dagger M_j|\psi\rangle = \sum_j M_j|\psi\rangle\langle\psi|M_j^\dagger.$$

Since $|\psi\rangle$ is arbitrary, we have $\sum_j M_j^\dagger M_j = I$. In general, suppose M_j acts on $\mathcal{H}_S$ such that $\sum_j M_j^\dagger M_j = I$. Then $\{M_j^\dagger M_j\}$ forms a POVM and

$$\rho_S \mapsto \sum_j M_j \rho_S M_j^\dagger$$

is a quantum operation.

9.3 Examples of Quantum Channels

Bit-Flip Channel

Define the bit-flip channel by

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_x \rho_S \sigma_x, \quad 0 \le p \le 1.$$

The input state ρ_S is bit-flipped $(|0\rangle, |1\rangle) \mapsto (|1\rangle, |0\rangle$ with a probability p, and remains unchanged with a probability $1-p$. The Choi/Kraus operators are: $E_0 = \sqrt{1-p}I$ and $E_1 = \sqrt{p}\sigma_x$.

This can be modeled by

$$\rho_S \mapsto V(\rho_S \otimes [(1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|])V^t = (1-p)\rho_S \otimes |0\rangle\langle 0| + p\,\sigma_x \rho_S \sigma_x |1\rangle\langle 1|$$

with $V = I \otimes |0\rangle\langle 0| + \sigma_x |1\rangle\langle 1|$ so that

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_x \rho_S \sigma_x.$$

We may also use $\rho_E = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle$.

Use the block sphere representation:

$$\rho_S = \frac{1}{2}\left(I + \sum_{k=x,y,z} c_k \sigma_k\right), \quad c_x^2 + c_y^2 + c_z^2 \le 1.$$

Then

$$\mathcal{E}(\rho_S) = \frac{1}{2}\left(I + c_x\sigma_x + (1-2p)c_y\sigma_y + (1-2p)c_z\sigma_z\right).$$

So, the block sphere shrinks in the y and z directions by a factor of $|1-2p|$.

Phase-Flip Channel

The phase-flip channel is defined by

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_z\rho_S\sigma_z, \qquad 0 \le p \le 1.$$

The input state ρ_S is phased-flipped $(|0\rangle, |1\rangle) \mapsto (|0\rangle, -|1\rangle$ with a probability p, remains unchanged with a probability $1-p$. The Choi/Kraus operators are: $E_0 = \sqrt{1-p}I$ and $E_1 = \sqrt{p}\sigma_z$.

This can be modeled by

$$\rho_S \mapsto V(\rho_S \otimes [(1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|])V^\dagger = (1-p)\rho_S \otimes |0\rangle\langle 0| + p\,\sigma_z\rho_S\sigma_z|1\rangle\langle 1|$$

with $V = I \otimes |0\rangle\langle 0| + \sigma_z|1\rangle\langle 1|$ so that taking partial trace (to remove the environment effect) will give

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + p\,\sigma_x\rho_S\sigma_x.$$

We may also use $\rho_E = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle$.

Use the block sphere representation:

$$\mathcal{E}(\rho_S) = \frac{1}{2}\left(I + (1-2p)c_x\sigma_x + (1-2p)c_y\sigma_y + c_z\sigma_z\right).$$

So, the block sphere shrinks in the y and z directions by a factor of $|1-2p|$.

Depolarizing Channel

The depolarizing channel is defined by

$$\mathcal{E}(\rho_S) = (1-p)\rho_S + pI/2, \qquad 0 \le p \le 1.$$

The input state ρ_S is sent to a maximally mixed state $I/2$ with a probability p, remains unchanged with a probability $1-p$. The Choi/Kraus operators are: $E_0 = \sqrt{1-3p/4}I$ and $E_k = \sqrt{p/4}\sigma_k$ for $k = x, y, z$. Let $p' = 3p/4$. Then

$$\mathcal{E}(\rho_S) = (1-p')\rho_S + (p'/3)\sum_k \sigma_k \rho_S \sigma_k.$$

This can be modeled by

$$\rho_S \mapsto V(\rho_S \otimes I/2 \otimes [(1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|])V^t = (1-p)\rho_S \otimes (I/2) \otimes |0\rangle\langle 0| + p\,(I/2) \otimes \rho_S |1\rangle\langle 1|$$

with $V = I_4 \otimes |0\rangle\langle 0| + U_{SWAP}|1\rangle\langle 1|$.

Use the block sphere representation:

$$\mathcal{E}(\rho_S) = \frac{1}{2}\left(I + (1-p)\sum_k c_k \sigma_k\right).$$

So, the block sphere shrinks in the $1x, y, z$ directions by a factor of $1-p$.

Amplitude-Damping Channel

The amplitude-damping channel is defined by

$$\mathcal{E}(\rho_S) = E_0 \rho_S E_0^\dagger + E_1 \rho_S E_1^\dagger, \qquad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \; E_1 = \sqrt{p} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It describes the decay of the qubit from $|1\rangle$ to $|0\rangle$ with probability p.

Use the block sphere representation:

$$\mathcal{E}(\rho_S) = \frac{1}{2} \left(I + \sqrt{1-p} c_x \sigma_x \sqrt{1-p} c_y \sigma_y + [p + (1-p) c_z] \sigma_z \right).$$

So, the block sphere shrinks in the x and y directions by a factor of $\sqrt{1-p}$, and shrinks in the z direction by a factor of $1-p$ and then shift to the north pole by p.

9.4 Lindblad Equation

Recall the Lioville-von Neumann equation for a closed system

$$\frac{\partial \rho_S}{\partial t} = \frac{1}{i}[H, \rho_S].$$

For an open system, it becomes

$$\rho_S(t) = \mathrm{Tr}_E(U(t)(\rho_S \otimes \rho_E)U(t)^\dagger].$$

Thus,

$$\frac{\partial \rho_S}{\partial t} = \frac{\partial \rho_S}{\partial t}\mathrm{Tr}_E(U(t)(\rho_S \otimes \rho_E)U(t)^\dagger] = \frac{1}{i}\mathrm{Tr}_E[H, U(t)(\rho_S \otimes \rho_E)U(t)^\dagger].$$

Assume that the system relies very little about previous history, the we use the **Markovian approximation**, and the equation will be of the form

$$\frac{\partial \rho_S}{\partial t} = \mathcal{L}\rho_S(t) = \frac{1}{i}[\tilde{H}, \rho_S] + \sum_{K=1}^{N} \gamma_K \left(L_k \rho_S L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho_S - \frac{1}{2}\rho_S L_k^\dagger L_k \right).$$

This is known as the Lindblad equation; the operators L_k are known as the Lindblad operators.