Math 430 Algebra II Exam 1 **Sample solution based on that of Liam Bench**

20.8 Using Theorem 20.3 every member of $F(a)$, where $F = \mathbb{Z}_2$, can be uniquely expressed as $c_2a^2+c_1a+c_0$ where $c_i \in \mathbb{Z}_2$. There are 2 choices for each c_i so $2^3 = 8$ different elements of $F(a)$.

	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
0	0	0	0	0	0	0	0	0
1	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
a	0	a	a^2	a^2+a	$a+1$	1	a^2+a+1	a^2+1
$a+1$	0	$a+1$	a^2+a	a^2+1	a^2+a+1	a^2	1	a
a^2	0	a^2	$a+1$	a^2+a+1	a^2+a	a	a^2+1	1
a^2+1	0	a^2+1	1	a^2	a	a^2+a+1	$a+1$	a^2+a
a^2+a	0	a^2+a	a^2+a+1	1	a^2+1	$a+1$	a	a^2
a^2+a+1	0	a^2+a+1	a^2+1	a	1	a^2+a	a^2	$a+1$

20.9. First we recognize $a^3+a+1 = 0$ and so $a^3 = a+1$. So $a^5 = a^2(a^3) = a^2(a+1) = a^3+a^2 = a^2+a+1$. Next $a(a^2+1) = a^3+a = 1$ and so $a^{-1} = a^2+1$. So $a^{-2} = (a^{-1})^2 = (a^2+1)^2 = a^4+1 = (a+1)a+1 = a^2+a+1$. Next we note that $a^7 = a(a+1)^2 = a(a^2+1) = a^3+a = 1$. So $a^{100} = a^2(a^7)^{14} = a^2(1^{14}) = a^2$

20.10. $(a^2)^3 + a^2 + a = a^6 + a^2 + 1 = (a^2+1) + (a^2+1) = 0$ and $(a^2+a)^3 + a^2 + a + 1 = (a^2+a)(a^4 + a^2) + a^2 + a + 1 = (a^2+a)(a) + a^2 + a + 1 = a^3 + a + 1 = (a+1) + a + 1 = 0$.

20.12. The elements of $F = \mathbb{Q}(\pi^3)$ are of the form $(a_n(\pi^3)^n + \cdots + a_1\pi^3 + a_0)/(b_m(\pi^3)^m + \cdots + b_1\pi^3 + b_0)$ with $a, b \in \mathbb{Q}$ and at least one $b_i \neq 0$. Consider $p(x) = (x^3 - \pi^3) \in F[x]$. If $p(x)$ is reducible, then $p(x)$ has a zero $\mu = (a_n(\pi^3)^n + \cdots + a_1\pi^3 + a_0)/(b_m(\pi^3)^m + \cdots + b_1\pi^3 + b_0)$ in F. But then $\mu^3 = \pi^3$ implies that

$$(a_n(\pi^3)^n + \cdots + a_1\pi^3 + a_0)^3 = \pi^3(b_m(\pi^3)^m + \cdots + b_1\pi^3 + b_0)^3.$$

Hence, π is the zero of polynomial over $\mathbb{Q}$, which is impossible. So, $p(x)$ is irreducible. We also know $p(\pi) = \pi^3 - \pi^3 = 0$. Using Theorem 20.3 we know $F[x]/\langle p(x)\rangle \approx F(\pi)$. So the members of $F(\pi)$ can be uniquely written as $c_2\pi^2 + c_1\pi + c_0$ with $c \in \mathbb{Q}(\pi^3)$. So $1, \pi, \pi^2$ is the basis of $F(\pi)$ over F.

20.18. $(3+4\sqrt{2})((-3)/(23) + (4)/(23)\sqrt{2}) = ((-9)/(23) + (32)/(23)) + ((12)/(23) - (12)/(23))\sqrt{2} = 1$ and so $(\frac{-3}{23} + \frac{4}{23}\sqrt{2}) = (3+4\sqrt{2})^{-1}$.