Algebra II Exam 2 Sample Solution based on that of Liam Bench

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42. Since a is a primitive element it is a generator of $\mathrm{GF}(5^n)^*$. The order of 2 is 4. So we want the smallest number k that makes $(a^k)^4 = 1$. Well the order of a generator is $5^n - 1$ so $a^{(5^n-1)/4} = 2$ or 3. If $a^{(5^n-1)/4} = 2$ then k is $(5^n - 1)/4$. If it is 3 then $k = 3(5^n - 1)/4$. So, the smallest integer k is $(5^n - 1)/4$.

Note: The number $(5^n - 1)/4 = (5^n - 1)/(5 - 1) = 5^{n-1} + 5^{n-2} + \cdots + 1$ is an integer.

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12. First note that R is commutative because it is contained in a field and has unity because it contains a field. We need only show that every nonzero element $a \in R$ has an inverse in R. Let a be a nonzero element of R. So $a \in E$ and is therefore algebraic over F. So there is a unique monic irreducible $p(x) \in F[x]$ such that $p(a) = 0$. Let $\deg p(x) = n$. Well $a^{-1} \in F(a)$ and every element of $F(a)$ can be written as $c_{n-1}a^{n-1} + ... + c_1 a + c_0$ by Theorem 20.3. So $a^{-1} = d_{n-1}a^{n-1} + ... + d_1 a + d_0 \in R$.

Alternatively, $0 = p(a) = b_0 + \cdots + a^n$ implies that $1 = ad = da$ with $d = -b_0^{-1}(b_1 + b_2 a + \cdots a^{n-1})$. So, $d = a^{-1}$.

Chapter 24

50. Clearly, $N(H) \subseteq N(N(H))$. To prove the converse, H is the only Sylow subroup in $N(H)$. If $y \in N(N(H))$, then $yHy^{-1} \subseteq yN(H)y^{-1} = N(H)$ is a Sylow subgroup in $N(H)$ so that $yHy^{-1} = H$. Hence, $y \in N(H)$.

52. Let H be a Sylow p-subgroup and K be a Sylow q-subgroup. So $|H| = p^2$ and $|K| = q^2$. Let n_p be the number of Sylow p-subgroups. So $n_p = 1 + kp$ and $n_p | q^2$. So $n_p = 1, q, \text{or} q^2$. If $n_p = q$ then $1 + kp = q$. So $q^2 - 1 = (1 + kp)^2 - 1 = k^2p^2 + 2kp$ and $p | q^2 - 1$. This is a contradiction. If $n_p = q^2$ then $1 + kp = q^2$ and $p | q^2 - 1$. This is also a contradiction so n_p must equal 1. So H is the only Sylow p-subgroup and therefore normal in G. By a symmetric argument K is also normal in G. $|HK| = |p^2q^2|$ so $G = HK \approx H \oplus K$. Since $|H| = p^2$ and $|K| = q^2$, H and K are both Abelian. So G is isomorphic to an external direct product of Abelian groups and therefore Abelian itself.

Three pairs of primes that satisfy this are $(5, 7), (7, 11), (11, 13)$.

60. Let $|G| = 45 = 3^2 \cdot 5$. Let n_3 be the number of Sylow 3-subgroups. So $n_3 = 1 + 3k$ and $n_3 | 5$. So $n_3 = 1$. So this group is unique. Call it H. We have $n_5 = 1 + 5k$ and $n_5 | 9$. So $n_5 = 1$. This group is also unique. Call it K. So H and K are unique and therefore normal in G. Since the orders of H and K are relatively prime we have $|HK| = 45 = |G|$. So $G = HK \approx H \oplus K$. K is of prime order so it is isomorphic to $\mathbb{Z}_5$. Also H is of a square prime order so it is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. So G is isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_5$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.