Chapter 19 Vector Spaces

A **vector space** V over a field $\mathbb{F}$ is an Abelian group $(V, +)$ with a scalar multiplication μv for any $\mu \in \mathbb{F}$ and $v \in V$ such that $1v = v$, $(ab)v = a(bv)$, $a(u+v) = au + av$ $(a+b)v = av + bv$ for any $a, b \in \mathbb{F}$ and $u, v \in V$.

Examples $\mathbb{F}^n$, $M_n(\mathbb{F})$, $\mathbb{F}[x]$.

Examples An extension field $\mathbb{E}$ over the ground field.

(a) $\mathbb{C}$ over $\mathbb{R}$. (b) $\mathbb{R}$ over $\mathbb{Q}$. (c) $\mathbb{Z}_p[x]/\langle f(x)\rangle$ over $\mathbb{Z}_p$.

- A set $S \subseteq V$ is linearly dependent if there is a nontrivial combination of a finite collection of vectors in S equal to 0.
- It is a basis if V if it is a spanning set of V.

Linear independence and basis

Theorem Every vector space has a basis.
If V has a basis with n elements, then every basis has n elements. In such a case, we say that V has dimension n. We use the convention that dimension V is 0 if $V = \{0\}$.

Question Can we say that two bases of a vector space must have the same cardinality? [A writing project?]

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