Chapter 23 Geometric Construction

Historical background

- Ancient Greeks want to do geometric construction using straight edge (without markings) and a compass.
- One can bisect an angle, construct an equilateral triangle, a square, a regular pentagon, a regular hexagon.
- Questions:

  Double a cube: Given a cube, construct a cube with double volume.

  Square a circle: Construct a square with the same area of a circular disk.
- One can transform the problems to algebraic problems.

Construction of regular polygons

- It was known in Euclid's time that one can construct a regular n-side polygon if n is $2^k, 2^k 3, 2^k 5, 2^k 3 \cdot 5$.
- Gauss in 1976 at the age of 19 do the construction of 17-side regular polygon.
- Gauss in 1801 asserted that an n-sided regular polygon is constructible if and only if $n = 2^k p_1 \cdots p_r$ such that p_i are distinct Fermat prime, i.e., prime of the form $F(s) = 2^{2^s} + 1$.
- Note that $F(0), F(1), F(2), F(3), F(4)$ are 3, 5, 17, 257, and 65537 are primes, but

$$F(5) = 2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641 \times 6700417.$$

- Some open questions.

  * Is $F(s)$ composite for all $s > 4$?
  * Are there infinitely many Fermat primes? (Eisenstein 1844.)
  * Are there infinitely many composite Fermat numbers?
  * Are there any Fermat number which is not square-free?

Constructible numbers

- If α, β are constructible, then so are $\alpha+\beta, \alpha-\beta, \alpha\beta$, and also α/β if $\beta \neq 0$.
- So, the set of constructible numbers is a field $\mathbb{F}$ in $\mathbb{R}$ containing $\mathbb{Q}$.
- We can construct lines and circles of the form

$$ax+by+c=0, \qquad x^2+y^2+ax+by+c=0, \qquad a,b,c \in \mathbb{F}.$$

- The geometric construction can can generate points (x, y) in $\mathbb{R}^2$ from a point by intersecting

  (1) two lines, (2) a line and a circle, (3) two circles,

  of the above types.
- Type (1) and (3) do not generate new points; type (2) intersection generate points of the form $\mathbb{F}(\sqrt{\alpha})$ with $\alpha \in \mathbb{F}$.

Inconstructability

- Of course, double the cube means constructing number of the form $2 = \alpha^3$. The splitting field $\mathbb{F}(2^{1/3})$, which is impossible.
- Trisection of $\theta = \pi/3$ is impossible. If yes, one can get $\cos(\theta/9)$, where

$$\cos\theta = 4\cos^3\theta - 3\cos\theta,$$

so that $\cos(\pi/9)$ is a zero of $8x^3 - 6x - 1$, which is irreducible, and the splitting field is not of degree 2^k.