Chapter 26 Generators and Relations

Motivation

Construct the largest group satisfying some prescribed properties.

For example: D_4 is the only group generated by a, b, satisfying

$$a^4 = b^2 = (ab)^2 = e.$$

One can show that any other group generated by two elements that satisfy the above relations is isomorphic to D_4.

The subgroup $\{R_0, R_{180}, H, V\}$ of D_4 is generated by $a = R_{180}$ and $b = H$ that satisfy

$$a^4 = b^2 = (ab)^2 = e \text{ and } a^2 = e.$$

Definitions and Notation

Let $S = \{a, b, c, \ldots\}$. Create $S^{-1} = \{a^{-1}, b^{-1}, c^{-1}, \ldots\}$.

Define the set $W(S)$ of words of finite length $x_1 \cdots x_k$ with $x_i \in S \cup S^{-1}$. Combine two words $x_1 \cdots x_k$ and $y_1 \cdots y_t$ by juxtaposition yielding $x_1 \cdots x_k y_1 \cdots y_t$, and let e represents the empty word.

Define an equivalence relation on $W(S)$ by: two words are equivalent if one can be obtained from the other by adding or deleting words of the form xx^{-1} or $x^{-1}x$, where $x \in S$.

Theorem 26.1 The set of equivalence classes of $W(S)$ under the above relation form a group under the operation $[u][v] = [uv]$.

The group is called a free group on S.

Theorem 26.2 Every group is a homomorphic image of a free group.

Proof. Let G be a group, and let S be a generating set. Then define $\phi : W(S)/\sim \to G$ by $\phi([x_1 \cdots x_k]) = (x_1 \cdots x_k)_G$ ...

Corollary Every group is isomorphic to a factor group of a free group.

Presentations: Generators and relations

Let G be a group generated by $A = \{a_1, \ldots, a_n\}$, and let F be the free group on A.

Let $W = \{w_1, \ldots, w_t\}$ be a subset of F and N be the smallest normal subgroup of F containing W.

Then G is given by the generators $a_1, \ldots, a_n$ and the relations $w_1, \ldots, w_t = e$ if there is an isomorphism $\phi : F/N \to G$ such that $\phi(a_i N) = a_i$.

In such a case, we write

$$G = \langle a_1, \ldots, a_n | w_1 = \cdots = w_t = e \rangle.$$

Example $\mathbb{Z} = \langle a \rangle$.

Example $D_4 = \langle a, b | a^4 = b^2 = (ab)^2 = e \rangle$.

Proof. Let F be the free group on $\{a, b\}$, and let N be the smallest subgroup containing $\{a^4, b^2, (ab)^2\}$.

Define $\phi : F \to D_4$ such that $\phi(a) = R_{90}, \phi(b) = H$. Then $N \subseteq \ker(\phi)$. Then $F/\ker(\phi)$ is isomorphic to D_4.

Claim: $F/N = K = \{N, aN, a^2N, a^3N, bN, abN, a^2bN, a^3bN\}$.

We need only show that $(aN)K = K$ and $(bN)K = K$. The first case is clear.

One need to focus on the second case. For example, $(bN)(aN) = baNb^2 = babNb = a^{-1}ababNb = a^{-1}Nb = a^{-1}a^4Nb = a^3Nb = s^3bN$.

Other cases are similar.

So, F/N has at most 8 elements.

Now, $F/\ker(\phi)$ is isomorphic to $(F/N)/(\ker\phi/N)$.

Thus, $\ker(\phi)/N$ is trivial, i.e., $\ker(\phi) = N$, and hence F/N the same as $F/\ker(\phi)$, which is isomorphic to D_4.

More results

Theorem 16.3 Let $G = \langle a_1, \ldots, a_n | w_1 = \cdots = w_t = e \rangle$, and
$\tilde{G} = \langle a_1, \ldots, a_n | w_1 = \cdots = w_t = w_{t+1} \cdots = w_{t+k} = e \rangle$.
Then $\tilde{G} = \phi[G]$ for some group homomorphism ϕ.

Proof. Exercise 5. □

Corollary If K is a group satisfying the defining relations of a finite group G (with the same set of generators) and $|K| \geq |G|$, then K is isomorphic to G.

More examples

Example Quaternions. $G = \langle a, b | a^2 = b^2 = (ab)^2 \rangle$. Let F be the free group on $\{a, b\}$ and N is the smallest normal subgroup containing $b^{-2}a^2, (ab)^{-2}a^2$.

Show that $K = N, bN, b^2N, b^3N, aN, abN, ab^2N, ab^3N\}$ is closed under multiplication.

Then show that, say, by inspecting the group table, $K \sim \{\pm 1, \pm i, \pm j, \pm k\}$ with $i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik$ satisfies the relations and has 8 elements.

Example $G = \langle a, b | a^3, b^9, a^{-1}ba^{-1}b^{-1} \rangle$ implies that $G = \mathbb{Z}_3$.

Note that $b^{-1} = a^{-1}ba$ imples that $b = a^{-1}b^{-1}a$. Then $b = a^{-3}ba^3 = a^{-2}b^{-1}a^2 = a^{-1}ba^1 = b^{-1}$. Thus, $b^2 = e$. Because $b^9 = e$. So, $b = e$, and $G = \mathbb{Z}_3$.

Theorem 26.4 Up to isomorphism, there are five groups of order 8: $\mathbb{Z}_8, \mathbb{Z}_4 \oplus \mathbb{Z}, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, D_4, the quaternions.

Proof. Suppose G is non-Abelian. There is an element a of order 4. Else all elements has order 2 and is Abelian.

Thus, $G = H \cup Hb$ with $H = \langle a \rangle$. Now, $b^2 \notin \{b, ab, a^2b, a^3b\}$. Else, $b \in H$.

Also, $b^2 \neq a$ because b^2 commute with b, but a does not.

Similarly, $b^2 \neq a^{-1} = a^3$.

So, $b^2 = e$ or a^2. In the former case, we get D_4; in the latter case, we get the quaternions. □

Characterization of Dihedral groups

Theorem 26.5 Any group generated by a pair of order 2 elements is dihedral.

Proof. Suppose $G = \langle a, b | a^2, b^2 \rangle$.

If (ab) has infinite order, then $F = \{e, a, b, ab, ba, aba, bab, abab, baba, \dots\}$.

If $G = F/H$ and $H \neq \{e\}$. Then H contains $(ab)^i, (ab)^i a, (ba)^i$, or $(ba)^i b$.

Then G cannot contain elements with word length larger than $2i + 2$.

Thus, G is finite and ab has finite order.

If (ab) has order n, then $G = \{a, b, ab, ba, \dots, (ab)^n = e = (ba)^n\}$. □