Chapter 29 Symmetry and Counting

Motivation

Basic Problem Counting different designs under (permutation) symmetry.

Example Color the vertexes of a hexagonal ceramic by 3 black and 3 white.

Rotation symmetry under C_6. There are four types.

Example Construct bracelet using 3 black and 3 white beads.

Symmetry under D_6. There are three types.

See http://people.math.sfu.ca/~jtmulhol/math302/notes/23-Burnside.pdf

Let G be a group of permutation acting on S.
For $\phi \in G$, let $Fix(\phi) = \{i \in S : \phi(i) = i\}$.
For $i \in G$, let $stab_G(i) = \{\phi \in G : \phi(i) = i\}$, and $orb_G(i) = \{\phi(i) : \phi \in G\}$.

Burnside's Theorem

Theorem 29.1 If G is a finite group of permutations on a set S, then the number of orbits of elements of S under G is $\frac{1}{G}\sum_{\phi \in G} |fix(\phi)|$.

Proof. Consider $(\phi, i) \in G \times S$ such that $\phi(i) = i$. The number of such pairs is:

$$n = \sum_{\phi \in G} |fix(\phi)| = \sum_{i \in S} |stab_G(i)|.$$

If s and t lie in the same orbit, then $orb_G(s) = orb_G(t)$, and

$$|stab_G(s)| = |G|/|orb_G(s)| = |G|/|orb_G(t)| = |stab_G(t)|.$$

So, if we choose $s \in S$ and sum over $orb_G(s)$, we have

$$\sum_{t \in orb_G(s)} |stab_G(t)| = |orb_G(s)||stab_G(s)| = |G|.$$

Thus, summing over all $g \in G$, one orbit at a time, we have

$$\sum_{\phi \in G} |fix(\phi)| = \sum_{i \in S} |stab(i)| = |G| \cdot (\text{number of orbits}).$$

Example

Example the number distinct colorings of the faces of a cube using 3 colors:

Let X be the set of 3^6 possible face colour combinations.

Let the rotation group G of the cube act on X with 24 elements.

- one identity element which leaves all 3^6 elements of X unchanged.
- six 90-degree face rotations, each of which leaves 3^3 of the elements of X unchanged.
- three 180-degree face rotations, each of which leaves 3^4 of the elements of X unchanged
- eight 120-degree vertex rotations, each of which leaves 3^2 of the elements of X unchanged
- six 180-degree edge rotations, each of which leaves 3^3 of the elements of X unchanged

The average fix size is thus $\frac{1}{24}\left(3^6 + 3^3 6 + 3^4 3 + 3^2 8 + 3^3 6\right) = 57.$

In general, if we use n colors, we have $\frac{1}{24}\left(n^6 + n^3 12 + n^4 3 + n^2 8\right)$. (Proof.)

Note that a key idea is the group action on a set.

G acts on G, G acts on coset of $H \leq G$, G acts on geometrical objects/patterns.