Math 430 Homework 1 Solution based on that of Liam Bench

16.3. The zeros are $1, 2, 4, 5$. Check: $f(1) = 1^2 + 3 \cdot 1 + 2 = 6 = 0$, $f(2) = 2^2 + 3 \cdot 2 + 2 = 12 = 0$, $f(4) = 4^2 + 3 \cdot 4 + 2 = 30 = 0$, $f(5) = 5^2 + 3 \cdot 5 + 2 = 42 = 0$.

16.4. Suppose char $R \neq$char $R[x]$

Case 1: char $R = 0$

So char $R[x] \neq 0$. So for some positive integer n, $nf(x) = 0, \forall f(x) \in R[x]$. All zero degree polynomials in $R[x]$ are the same as elements in R. So $ny = 0, \forall y \in R$. This a contradiction and char $R =$char $R[x]$.

Case 2: char $R = n > 0$

char $R[x] = m < n$ because $nf(x) = na_0 + na_1 x + \dots na_m x^m = 0$ and $na_i = 0, a_i \in R$ and char $R \neq$char $R[x]$. Take any zero degree $f(x) = a \in R[x]$ (also $a \in R$). So $ma = mf(x) = 0, \forall a \in R$, a contradiction as n was chosen to be the least that fulfills this. So $R =$char $R[x]$.

Remark One can use the fact that in a commutative ring with 1, the characteristic is determined by the smallest positive integer m such that $m1 = 0$, or $m1 \neq 0$ for any positive integer m.

16.6 All degree two polynomails are They are $x^2, x^2 + 1, x^2 + x, x^2 + x + 1$. Note that two polynomials $f(x)$ and $g(x)$ are identical functions if $f(x) - g(x) = 0$ for all $x = 0, 1$. Thus, $f(x) - g(x)$ must have zeros $0, 1$ and has degree at least 2. So, no two degree 2 polynomials $f(x)$ and $g(x)$ in $\mathbb{Z}_2[x]$ have such properties.

16.13. $f(x) = g(x)(4x^2 + 3x + 6) + (6x + 2)$.

16.15. $(2x + 1)$ is its own inverse. Check: $(2x + 1)(2x + 1) = 4x^2 + 4x + 1 = 1$.

16.18. In fact, we can show that we can show that if R has zero divisor, there must be a degree 1 polynomial with two zeros. Suppose R has zero divisor $ab = 0$ such that a, b are nonzero. Then $f(x) = ax$ is degree 1 with at least two zeros 0 and b. So, Corollary 3 will not hold.

16.20. Let B be an ideal and $B \neq \langle x \rangle$ and $B \supseteq \langle x \rangle$. So $\exists g(x) \in B$ such that $g(x) \notin \langle x \rangle$. Then $g(x)$ has nonzero constant term α, else, $g(x) \in \langle x \rangle$. Then we can find $h(x) \in \langle x \rangle$ such that $\alpha = g(x) - h(x) \in B$. If follows that $\mathbb{Q}[x] = \langle \alpha \rangle \subseteq B$.

16.24. Consider the polynomial $f(x) - g(x)$. Since $f(a) = g(a)$ for infinitely many a, $f(a) - g(a) = 0$ for those a. Using the fact a polynomial of degree n over a field has at most n zeros, counting multiplicities, $f(x) - g(x) = 0$ as 0 is the only polynomial with infinitely many zeros. So $f(x) = g(x)$.

16.60. By the factor theorem, we need to compute $f(-3) = (-3)^{51}$. By Fermat's little theorem, $x^7 = x$ for all nonzero $x \in \mathbb{Z}_7$. So, $(-3)^{49+2} = (-3)^{49}(-3)^2 = (-3)^3 = 6$.

16.34. Note that $f(x) \in \mathbb{Z}_3[x]$ satisfies $f(a) = 0$ for $a = 0, 1, 2$, if and only if $f(x)$ has linear factors $x(x - 1)(x - 2)$. Thus, every $g(x) \in \langle x(x - 1)(x - 2) \rangle$ has the desired property.