Math 430 Homework 3 Solution based on that of Liam Bench

18.10 10. Let D be a PID We show that $p \in D$ is reducible if and only if $\langle p \rangle$ is not maximal.

Proof. Suppose $A = \langle p \rangle$ is not maximal. Then there is an ideal B containing A properly and not equal to D. Suppose $B = \langle q \rangle$. Then $p = qr$. If q is a unit, then $B = D$, if r is a unit, then $A = B$. Thus, $p = qr$ is reducible. Conversely, if $p = qr$ is reducible so that neither q nor r is a unit, then $B = \langle q \rangle$ properly contains A and not equal D.

18.12. Let D be a principal ideal domain and I be a proper ideal of D. So $I = \langle p \rangle$. Take q as irreducible factor of p. (D is also a UFD) So $p = qt$ and $\langle p \rangle \subset \langle q \rangle$ and $\langle q \rangle$ is maximal as proved in problem 10.

18.16. Consider $\mathbb{Z}[2i] \subset \mathbb{Z}[i]$. According to example 7 $\mathbb{Z}[i]$ is a Euclidean domain and therefore a UFD. It is routine to check that $\mathbb{Z}[2i]$ is a subring with 1 and without zero divisors. So, $\mathbb{Z}[2i]$ is a subdomain. Now, $4 = 2 \cdot 2 = (2i)(-2i) \in \mathbb{Z}[2i]$. If 2 is reducible then $2 = xy$ and $4 = N(2) = N(x)N(y)$. So $N(x) = 2$ and $2 = a^2 + b^2$ which does not have a solution of the form $a + 2bi \in \mathbb{Z}[2i]$. So 2 is irreducible. Similarly, suppose $2i$ is reducible. So $2i = xy$ and $4 = N(2i) = N(x)N(y)$. So $N(x) = 2$ which does not have a solution. So $2i$ and $-2i$ are irreducible. Finally 2 and $2i$ are not associates because $i \notin \mathbb{Z}[2i]$. Thus, $\mathbb{Z}[i]$ is not a UFD.

18.18. First $N(7) = 7^2$ so it is not prime. Suppose 7 is reducible. So $7 = xy$ and $49 = N(7) = N(x)N(y)$ and $N(x) = 7$. So $7 = a^2 - b^2 6$ so that $a^2 + b^2 \equiv 0$ in $\mathbb{Z}_7$. Now, $x \in \mathbb{Z}_7$ implies $x^2 \in \{0, 1, 2, 4\}$. So, we have $a \equiv b \equiv 0 \in \mathbb{Z}_7$. But then $7 = a^2 - b^2 6$ is divisible by 49, a contradiction.

18.20. In $\mathbb{Z}[\sqrt{-3}]$, $4 = 2^2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$. If 2 is reducible then $2 = xy$ and $4 = N(2) = N(x)N(y)$. So $N(x) = 2$ and $2 = a^2 + 3b^2$ which does not have a solution. So 2 is irreducible. If $1 + \sqrt{-3}$ is reducible then $1 + \sqrt{-3} = xy$ and $4 = N(1 + \sqrt{-3}) = N(x)N(y)$. So $N(x) = 2$ which does not have a solution. So $1 + \sqrt{-3}$ is irreducible and by a similar argument so is $1 - \sqrt{-3}$. So 4 does not have a unique factorization. So $\mathbb{Z}[\sqrt{-3}]$ is not a UFD and is there fore not a PID.

18.26. Note that $N((3 + 2\sqrt{2})^n) = (N(3 + 2\sqrt{2}))^n = (9 - (4 \cdot 2))^n = 1^n = 1$. Since $N(x) = 1$ if and only if x is a unit these numbers must be units.

18.28. In $\mathbb{Z}_{12}$, if $a, b \notin \{0, 2, 4, 6, 8, 10\}$, then $ab \neq \{0, 2, 4, 6, 8, 10\}$. So, $2|(ab)$ implies $2|a$ or $2|b$. Also, if $a, b \notin \{0, 3, 6, 9\}$, then $ab \neq \{0, 3, 6, 9\}$. So, $3|(ab)$ implies $3|a$ or $3|b$.

Note that the units in $\mathbb{Z}_{12}$ are: $1, 5, 7, 11$. If $2 = ab$ is reducible, then $2|a$ or $2|b$, and $4 \nmid (ab)$. So, we may assume $a \in \{2, 6, 10\}$ and $b \in \{3, 9\}$. But $2 \neq ab$ for any such choices.

On the other hand, $3 = 3 \cdot 9$ and 3 and 9 are not units so 3 is reducible.

18.34. Expanding for both pairs and reducing $\bmod 5$ gives us $3x^2 + 4x + 3$. Note that $4(3x + 2) = 2x + 3$ and $4(x + 4) = 4x + 1$ so both pairs are associates. So "two" factorization are the same up to permutation and associates.

Optional. Will solve it using the result in Chapter 21.