Algebra II Homework 4 **Sample solution based on that of Liam Bench**

20.2. We know $\sqrt{2}+\sqrt{3}\in\mathbb{Q}(\sqrt{2},\sqrt{3})$ because the elements of $\mathbb{Q}(\sqrt{2},\sqrt{3})$ are of the form $a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3}$, where $a,b,c,d\in\mathbb{Q}$. So $\mathbb{Q}(\sqrt{2}+\sqrt{3})\subseteq\mathbb{Q}(\sqrt{2},\sqrt{3})$. To show $\mathbb{Q}(\sqrt{2},\sqrt{3})\subseteq\mathbb{Q}(\sqrt{2}+\sqrt{3})$ it is sufficient to show $\sqrt{2},\sqrt{3}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$. Well $(\sqrt{2}+\sqrt{3})(\sqrt{3}-\sqrt{2})=1$ and so $(\sqrt{2}+\sqrt{3})^{-1}=\sqrt{3}-\sqrt{2}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$. So $\sqrt{2}+\sqrt{3}+(\sqrt{3}-\sqrt{2})=2\sqrt{3}$ and $\sqrt{3}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$. Also $-(\sqrt{3}-\sqrt{2})=\sqrt{2}-\sqrt{3}$. So $(\sqrt{2}+\sqrt{3})+(\sqrt{2}-\sqrt{3})=2\sqrt{2}$ and $\sqrt{2}\in\mathbb{Q}(\sqrt{2}+\sqrt{3})$. Therefore $\mathbb{Q}(\sqrt{2},\sqrt{3})=\mathbb{Q}(\sqrt{2}+\sqrt{3})$.

20.4. Note that $x^4+1=(x-a)(x-a^3)(x-a^5)(x-a^7)$ with $a=e^{i\pi/4}=(1+i)/\sqrt{2}$. The splitting field of the polynomial is $\mathbb{Q}(a,a^3,a^5,a^7)=\mathbb{Q}(a,a^3)=\mathbb{Q}(i,\sqrt{2}$ as $a=-a^5,a^3=-a^7$.

20.14. The only automorphism of $\mathbb{Q}(5^{\frac{1}{3}})$ is the identity automorphism. We know that $\phi(1)=1$ and $\phi(0)=0$. Let $n\in\mathbb{N}$ and $n\neq 0$. We have $\phi(n)=n\phi(1)=n$ and $n\phi(\frac{1}{n})=\phi(\frac{n}{n})=\phi(1)=1=n\cdot\frac{1}{n}$. So $\phi(\frac{1}{n})=\frac{1}{n}$. Hence, for any $\frac{q}{p}\in\mathbb{Q}$, we have $\phi(\frac{q}{p})=\phi(q)\phi(\frac{1}{p})=\frac{q}{p}$.

By properties of fields $\phi(5^{\frac{1}{3}})$ will determine the rest of the automorphism. Let $y=\phi(5^{\frac{1}{3}})$. Because $5=\phi(5)=\phi((5^{\frac{1}{3}})^3)=\phi(5^{\frac{1}{3}})^3=y^3$, we have $y=5^{\frac{1}{3}}$ and this is the identity automorphism.

20.20. We can say $ac+b\in F(c)$ because of field properties. So $F(ac+b)\subseteq F(c)$. To show $F(c)\subseteq F(ac+b)$ I show that $c\in F(ac+b)$. Well $c=a^{-1}((ac+b)-b)$ and $c\in F(ac+b)$. So $F(c)=F(ac+b)$.

20.22. Since $f(x)$ and $g(x)$ are relatively prime in $F[x]$ we can write $1=f(x)q(x)+g(x)p(x)$ with $q(x),p(x)\in F[x]$. This is due to the fact that $F[x]$ is a Euclidean domain. Well $1=f(x)q(x)+g(x)p(x)$ is also true in $K[x]$ and so they are relatively prime in $K[x]$.

20.28. I show that the quotient field $\mathbb{Z}_p(x)=\{f(x)/g(x)\in\mathbb{Z}_p[x],g(x)\neq 0\}$ has characteristic p and is not perfect. Let $f(x)=a_nx^n+\cdots+a_0$. So $pf(x)/g(x)=(pa_nx^n+\cdots+pa_0)/g(x)=0$. The characteristic is not less than p because it would imply the characteristic of Z_p is lower that p. Suppose for contradiction that $Z_p(x)$ is perfect. So $x=(f(x)/g(x))^p$ and therefore $xg(x)^p=f(x)^p$. If $\deg(f(x))=n$ and $\deg(g(x))=m$ we have $\deg(xg(x)^p)=mp+1=np=\deg(f(x)^p)$. So $(n-m)p=1$ which is impossible. So the field is not perfect.

20.32. If $f(x)=x^{21}+2x^9+1$ then $f'(x)=21x^{20}+18x^8=0$ in $\mathbb{Z}_3$. Since $0=0*(x^{21}+2x^9+1)$, $f(x)$ is a factor of $f'(x)$. So by Theorem 20.5 $f(x)$ has a multiple zero in some extension field.

Remark In fact, $f(x)=(x^7+2x+1)^3$.

20.38. In $\mathbb{Q}$, x^4-6x^2-7 factors into the irreducibles $(x^2+1)(x^2-7)$. This splits in $\mathbb{Q}(i,\sqrt{7})$ as $(x+i)(x-i)(x+\sqrt{7})(x-\sqrt{7})$. So $\mathbb{Q}(i,-i,\sqrt{7},-\sqrt{7})=\mathbb{Q}(i,\sqrt{7})$ is the splitting field.

20.40. Let $f(x)$ be an irreducible polynomial over a field F and $\deg f(x)=n$. We know $f(x)$ splits in some field into q distinct linear factors. So $f(x)$ has q distinct zeros each of multiplicity m. So we can write $f(x)=a(x-a_1)^m(x-a_2)^m\ldots(x-a_q)^m=ax^{mq}+\ldots$. So $\deg f(x)=n=mq$ and $q|n$.