Math 430 Algebra II Homework 5 Sample solution based on that of Liam Bench

21.3. Note that $\mathbb{F} = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \dots)$ has a basis of the form $\mathcal{B} = \{1\} \cup \{2^{m/n} : 1 \le m < n\}$. So, every $a \in \mathbb{F}$ is a finite combination of elements in $\mathcal{B}$, and $F(a)$ is a finite extension, and hence an algebraic extension. So, a is algebraic. Since it is true for all $a \in \mathbb{F}$, we see that $\mathbb{F}$ is algebraic.

Suppose $[\mathbb{F} : \mathbb{Q}] = n$ is finite. Let $m > n$. Then $\mathbb{Q}(2^{1/m}$ is a subspace of $\mathbb{F}$ over $\mathbb{Q}$ with dimension m, a contradiction.

21.6. Let $f(x), g(x)$ be irreducible over $\mathbb{F}$, $\deg f = n$ and $\deg g = m$ with $\gcd(n, m) = 1$, and $f(a) = 0$, $g(b) = 0$ for some a, b in extensions of $\mathbb{F}$. Note that $[\mathbb{F}(a, b) : \mathbb{F}] = [\mathbb{F}(a, b) : \mathbb{F}(a)][\mathbb{F}(a), F]$ so that $[\mathbb{F}(a, b) : \mathbb{F}]$ is divisible by n; $[\mathbb{F}(a, b) : \mathbb{F}] = [\mathbb{F}(a, b) : \mathbb{F}(b)][\mathbb{F}(b), F]$ so that $[\mathbb{F}(a, b) : \mathbb{F}]$ is divisible by m. Consequently, $[\mathbb{F}(a, b) : \mathbb{F}]$ is divisible by mn.

If $g(x) \in \mathbb{F}(a)[x]$ is reducible, and b is the zero of a irreducible factor $g_1(x)$ of $g(x)$ in $\mathbb{F}(a)[x]$ such that degree of $g_1(x)$ equals $m_1 < m$. Then $[\mathbb{F}(a, b) : \mathbb{F}] = [\mathbb{F}(a, b) : \mathbb{F}(a)][\mathbb{F}(a), F] = m_1 n < mn$, which is a contradiction.

21.8. By Example 6, $\mathbb{F} = \mathbb{Q}(\sqrt{3}, \sqrt{5}) = \mathbb{Q}(\sqrt{3} + \sqrt{5})$ satisfies $[\mathbb{F} : \mathbb{Q}] = 4$. Now, $\mathbb{Q}(\sqrt{15}) \subseteq \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Because $4 = [\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}(\sqrt{15}][\mathbb{Q}(\sqrt{15} : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}(\sqrt{15}]2$, we see that $[\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}(\sqrt{15}] = 2$, and $\{1, \sqrt{3} + \sqrt{5}\}$ is a basis for $\mathbb{Q}(\sqrt{3} + \sqrt{5})$ over $\mathbb{Q}(\sqrt{15})$.

For the second part note that $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2})$ because $\sqrt{2} = (\sqrt[4]{2})^2$. Next $\big[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}) : \mathbb{Q}\big] = \big[\mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})\big]\big[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}\big]$ and $\big[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}\big] = 4$. Let $\big[\mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{2}) : \mathbb{Q}\big] = n$. Since 4 and $\big[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}\big] = 3$ are relatively prime n is at least 12. But since $\big[\mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})\big]$ is at most 3, as $(\sqrt[3]{2})^3 - 2 = 0$, n is as most 12. So $n = 12$.

21.16. The minimal polynomial is $x^3 - 6x - 6$. First note that $2^{1/3} + 2^{2/3}$ is a solution to the equation. If there was a quadratic with $2^{1/3} + 2^{2/3}$ as a zero we would have $a(4 + 2 \cdot 2^{1/3} + 2^{2/3}) + b(2^{1/3} + 2^{2/3}) + c = 0$. There are not any nonzero a, b, c that fulfill this. So the degree 3 polynomial has minimal degree.

21.18. Up to isomorphism, we may assume that $\mathbb{E} \subseteq \mathbb{C}$. Let $\{1, \mu\}$ be a basis of $\mathbb{E}$ over $\mathbb{Q}$. Then μ is the zero of a degree 2 polynomial. We may assume that the polynomial $f(x) = ax^2 + bx + c$ has integer coefficients. Then the $\mu = (-b \pm \sqrt{b^2 - 4ac})/2a = q_1 + q_2\sqrt{d}$ for some $q_1, q_2 \in \mathbb{Q}$ and d is an integer not divisible by p^2 for any prime p. So, $\mathbb{E} = \mathbb{Q}(q_1 + q_2\sqrt{d}) = \mathbb{Q}(\sqrt{d})$.

20.20. Let $\big[\mathbb{E} : \mathbb{F}\big]$ be finite and $\{1, a_1, \dots, a_m\}$ be a basis of $\mathbb{E}$ over $\mathbb{F}$. Then $\mathbb{E} = \mathbb{F}(a_1, \dots, a_m)$. Here we may remove redundant terms if desired.

For the second part just reverse the argument. Extending $\mathbb{F}(a_1, a_2, ...a_i)$ to $\mathbb{F}(a_1, a_2, ...a_{i+1})$ increases the degree by a certain number. After adding all the a, get the number $\big[E : \mathbb{F}\big]$ with a finite number of factors.

20.24. In $\mathbb{Z}_3$, $x^4 - x^2 - 2 = x^4 + 2x^2 + = (x^2 + 1)(x^2 + 1)$. In $\mathbb{Z}_3(\sqrt{2})$, this becomes $(x - \sqrt{2})^2(x + \sqrt{2})^2$. So $\mathbb{Z}_3(\sqrt{2}) = \frac{\mathbb{Z}_3[x]}{\langle x^2+1 \rangle}$ is the splitting field for $x^4 - x^2 - 2$ in $\mathbb{Z}_3$.

20. 26. By the quadratic formula the roots of $x^2 + x + 1$ are $a = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$. So $i\sqrt{3} \in \mathbb{Q}(a)$. Well $(\frac{1}{2} + \frac{i\sqrt{3}}{2}) \in \mathbb{Q}(a)$ and $(\frac{1}{2} + \frac{i\sqrt{3}}{2})^2 = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$. So $\sqrt{a} \in \mathbb{Q}(a)$ and $\mathbb{Q}\sqrt{a} = \mathbb{Q}(a)$. A similar argument is true for the other root.

20.32. Let $\deg f(x) = n$ and $\deg g(x) = m$ and suppose $f(x)$ is irreducible over $\mathbb{F}(b)$. So $\big[\mathbb{F}(a, b) : \mathbb{F}(b)\big] = n$. Well $\big[\mathbb{F}(a, b) : \mathbb{F}\big] = \big[\mathbb{F}(a, b) : \mathbb{F}(b)\big]\big[\mathbb{F}(b) : \mathbb{F}\big] = nm$. If $g(x)$ was reducible over $\mathbb{F}(a)$ then $\big[\mathbb{F}(a, b) : \mathbb{F}(a)\big] < m$ which is impossible since $mn = \big[\mathbb{F}(a, b) : \mathbb{F}(a)\big]\big[\mathbb{F}(a) : \mathbb{F}\big] = \big[\mathbb{F}(a, b) : \mathbb{F}(a)\big]n$. So $g(x)$ is irreducible over $\mathbb{F}(a)$. The other direction is true by symmetry.

20.38. If $\mathbb{C}$ was the splitting field of a polynomial, $f(x)$, in $\mathbb{Q}$ we can write $\mathbb{C} = \mathbb{Q}(a_1, a_2, ..., a_n)$ where the a_i's are the roots of $f(x)$. So then $\mathbb{C}$ is a finite extension of $\mathbb{Q}$. But $\mathbb{C} \supseteq \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, ...)$ and $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, ...)$ is not a finite extension as proved earlier. So $\mathbb{C}$ is not a finite extension, and therefore not a splitting field of $f(x)$ over $\mathbb{Q}$.