**Algebra II Homework 7** **Solution based on that of Liam Bench**

**Chapter 24**

**5.** Consider the function $T$ which sends the coset $xN(k)$ to the conjugate $xKx^{-1}$ of $K$. So $T(xN(K)) = xKx^{-1} = yKy^{-1} = T(yN(K))$ if and only if $(y^{-1}xN(K)(x^{-1}y) = N(K)$, i.e., $y^{-1}x \in N(x)$. Hence, $T$ is well-defined, and is one-one. For any $xKx^{-1}$, we have $T(xN(k)) = xKx^{-1}$. So, $T$ is surjective. Hence, the number of conjugates is $|G : N(K)|$.

**6.** If $H$ is normal then it is the only conjugate of itself in $G$. But $G$ is not $H$ because $H$ is proper. So $G$ is not the union of the conjugates of $H$ if $H$ is normal.

Suppose $H$ is not normal and let $k$ be the number of conjugates of $H$ in $G$. So $k = |G : N(H)|$ by problem 5 and we know $|G|/|N(H)| = |G : N(H)|$. So $|G| = k|N(H)|$. Since $H \leq N(H)$ we have $k|N(H)| \geq k|H|$. Also $k|H| > |H_1 \cup ... \cup H_k|$ because each of the conjugates at least have the identity in common. So $|G| = k|N(H)| \geq k|H| > |H_1 \cup ... \cup H_k|$. So $G$ cannot be the union of the conjugates of $H$.

**8.** Let $G$ be a non-Abelian group of order $39 = 3 \cdot 13$. So $|Z(G)| = 1, 3 \text{or} 13$. If it is 3 or 13 then $|G/Z(G)|$ is prime and therefore cyclic. But then $G$ is Abelian. So $|Z(G)| = 1$.

So $39 = 1 + 3x + 13y$. So $2 \equiv y \mod 3$ and $y \leq 2$ So $y = 2$ and $x = 4$. So $39 = 1 + (4)3 + (2)13$ is the class equation.

Now let $G$ be a non-Abelian group of order $55 = 5 \cdot 11$. By the argument above, $|Z(G)| = 1$. So $54 = 5x + 11y$. So $4 \equiv y \mod 5$ and $y \leq 4$. So $y = 4$ and $x = 2$. So $55 = 1 + (2)5 + (4)11$ is the class equation.

**18.** Let $G$ be a group of order $56 = 2^3 \cdot 7$. The number of Sylow 2-subgroups is $n_2 = 1 \mod 2$ where $n_2|7$. So $n_2 = 1 \text{or} 7$. The number of Sylow 7-subgroups is $n_7 = 1 \mod 7$ where $n_7|8$. So $n_7 = 1 \text{or} 8$. If $n_2 = 1$ or $n_7 = 1$ then the corresponding Sylow subgroup is unique and therefore normal.

Suppose $n_2 = 7$ and $n_7 = 8$. The 8 Sylow 7-subgroups only have the identity in common. If two shared an element then they would be the same because every non-identity element generates the group. This is because they have prime order. So the Sylow 7-subgroups have $1+8\cdot 6 = 49$ elements. A Sylow 2-subgroup of order 8 does not share any elements other than the identity with they Sylow 7-subgroups. This is because the order of their elements is different. So adding 1 Sylow 2-subgroup gives us $49 + 7 = 56$ elements. The other Sylow 2-subgroups each add at least one element because they are separate groups. So there are more than 56 elements which is a contradiction. So either $n_2 = 1$ or $n_7 = 1$ and there is a proper nontrivial normal subgroup.

**20.** First note there is only one Sylow 7-subgroup because $n_7 = 1 \mod 7$ where $n_7|3$. This Sylow 7-subgroup, call it $H$, is therefore normal. The number of Sylow 3-subgroups is $n_3 = 1 \mod 3$ where $n_3|7$. So $n_3 = 1, \text{or} 7$. If there is one Sylow 3-subgroup, call it $K$, then it is normal. So $HK$ has 21 elements and $G \approx HK \approx \mathbb{Z}_7 \oplus \mathbb{Z}_3$, which is cyclic because 3 and 7 are relatively prime. But $G$ is not cyclic so there must be 7 Sylow 3-subgroups.

**26.** Let $G$ be a group of order $175 = 5^2 \cdot 7$. So $n_5 = 1 \mod 5$ where $n_5|7$. So $n_5 = 1$. Also $n_7 = 1 \mod 7$ where $n_7|25$. Thus, $n_7 = 1$. Let $H$ be the Sylow 5-subgroup and $K$ be the Sylow 7-subgroup. Then $H$ and $K$ are normal. So $G \approx HK$ and $H \cup K = \{e\}$. Now, for any $h \in H, k \in K$, we have $hkh^{-1}k^{-1} \in H \cap K = \{e\}$. So, $hk = kh$ so that $h_1k_1h_2k_2 = h_1h_2k_1k_2$, and thus $HK$ is Abelian. (Actually, $HK$ is isomorphic to the external direct product of $H$ and $K$. So $G \approx \mathbb{Z}_{25} \oplus \mathbb{Z}_7 \text{or} \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$. So $G$ is a product of cyclic groups of prime power order and therefore Abelian.)

**Problem from Class Notes**

**1.** Let $T$ be the map that maps the coset $\phi\text{stab}_G(i)$ to $\phi(i)$. Note that $\alpha\text{stab}_G(i) = \beta\text{stab}_G(i)$ if and only if $\alpha^{-1}\beta \in \text{stab}_G(i)$, i.e., $\beta(i) = \alpha(i)$. Thus, $T$ is well defined and one-to-one. To show it is onto let $k \in \text{orb}_G(i)$. So $\alpha(i) = k$ for some $\alpha \in G$ and $T(\alpha\text{stab}_G(i) = \alpha(i) = k$. So $T$ is onto. As a result, there is a bijection between the number of cosets of $\text{stab}_G(i)$, which is $|G|/|\text{stab}_G(i)|$ and $\text{orb}_G(i)$.

**2.** a) Let $h_1k_1, h_2k_2 \in HK_i$. So $h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2 = h_1k^*h_2^{-1}$. Because $H < N(K_i)$ we have $hK_i = K_ih$ and therefore $h_1k^*h_2^{-1} = h_1h_2^{-1}\hat{k} \in HK_i$. So $HK_i$ is a subgroup of $N(K_i)$ as $H, K_i < N(K_i)$.

Theorem 7.2 in the book gives us the second part.

b) The equation form part a) gives us $(|H||K_i|)/|HK_i| = |H \cap K_i|$. First $|H||K_i| = p^{2l}$. Second $|HK_i|$ has at least $p^l$ elements because it contains $H$. It is a power of $p$ because it divides $p^{2l}$. It is not greater than $p^l$ because $p^{l+1} \nmid |G|$. So $|H \cap K_i| = p^l$ and $|H| = |H \cap K_i|$. Also $|K_i| = p^l$ so $H \subseteq K_i$.

**3.** Let $a, b \in N(H)$. Let $ab^{-1}hba^{-1} \in ab^{-1}Hba^{-1}$. Now, $b^{-1}Hb = H$ and so $ab^{-1}hba^{-1} = a\hat{h}a^{-1} \in aHa^{-1} = H$. So $ab^{-1}Hba^{-1} \subseteq H$ and $N(H)$ is a subgroup.

The second part is proven in Problem 5.