Math 430 Homework 8 **Sample solution based on that of Liam Bench**

Chapter 25

8. Suppose G is a simple group of order $315 = 3^2 \cdot 5 \cdot 7$. Consider the number of Sylow 5-subgroups, n_5. $n_5 = 1 + 5k$ and divides 63. If $n_5 \neq 1$ then L_5, a Sylow 5-subgroup, is normal. So $n_5 = 21$. By problem 5 in chapter 24, this is $|G : N(L_5)|$. So $|N(L_5)| = |G|/|G : N(L_5)| = 15 = 3 \cdot 5$. So Theorem 24.6 tells us that $N(L_5)$ is cyclic. So $N(L_5)$ has an order 15 element and so does G.

Next consider the number of Sylow 3-subgroups, n_3. So $n_3 = 1 + 3k$ and divides 35. So $n_3 = 7$. So $|G : N(L_3)| = 7$. By the Embedding Theorem, G is isomorphic to a subgroup of A_7. So there is an order 15 element in A_7. Writing this permutation as a product of disjoint cycles it is either has a 15 cycle, which is impossible with 7 symbols, or a 3 and a 5 cycle, which requires at least 8 symbols. This is a contradiction so G is not simple.

14. Case 1. $p = q = r$. So $|G| = p^3$ and by Theorem 24.2 $Z(G)$ has more than one element. If $Z(G) = G$, therefore G is Abelian. The only Abelian simple groups are $\mathbb{Z}_n$ where $n = 1$ or n is prime. So G is not simple. Otherwise $Z(G)$ is a proper normal subgroup of G and therefore G is not simple.

Case 2. $p = q \neq r$. So $|G| = p^2 r$. Suppose G is simple. So $n_p = 1 + kp$ and $n_p | r$. If $n_p = 1$, then this Sylow p-subgroup is normal. So $n_p = r$. Next n_r is p or p^2 by the above reasoning. So $n_r = 1 + kr > r$. If $n_r = p$ then $p = n_r = 1 + kr > r = n_p > p$. This is a contradiction. So $n_r = p^2$. All of the Sylow r-subgroups have $r - 1$ elements of order r. So there are $p^2(r - 1)$ elements of order r in G. A Sylow p-subgroup has p^2 elements. So adding these elements we have $p^2(r - 1) + p^2 = p^2 r$ elements in G not including elements from the other Sylow p-subgroups. But G only has $p^2 r$ elements. This is a contradiction and so G is not simple.

Case 3. $p > q > r$. Suppose G is simple. So $n_p = 1 + kp$ and $n_p | qr$. Because $p > q > r$ and the fact that $n_p = 1$ would make this Sylow group normal, we have $n_p = qr$. Next $n_q | pr$. By the above reasoning either $n_q = p$ or $n_q = pr$. Every Sylow q-subgroup has $q - 1$ elements of order q. So there are at least $p(q-1)$ elements of order G. There are $qr(p-1)$ elements of order p. Because $p(q - 1) > r(q - 1)$, there are another $pqr - qr(p - 1) - r(q - 1) = r$ elements. But there are more than r elements we have to account for because there is more than 1 Sylow r-subgroup. This is a contradiction and G is not simple.

16. Suppose H is a subgroup of S_5 of order 40. So either all or half of H are even permutations. Suppose all of them are even. So H is a subgroup of A_5 of order 40. This is impossible because A_5 has order 60. Suppose half of the elements are even permutations. So the even permutations in H form a subgroup of A_5 of order 20. 60 does not divide the factorial of the index of this group, $(60/20)! = 3!$. So A_5 is not simple which is a contradiction. A similar argument holds if S_5 has a subgroup of order 30 showing that A_5 is not simple. So S_5 does not have a subgroup of order 30.

20. If $|G| = p^k$, the result follows form Sylow Theorem. If $q||G|$ with $q > p$ for some other prime, then $|G| \not| p!$, a contradiction. So, G is not simple.

24. First note that we can write any transposition, $(i, i+1)$, as $(i, i+1) = (12345)^{i-1}(12)(12345)^{5-(i-1)}$. Next we can think of any 2-cycle as a product of transpositions. A 2-cycle permutes 2 elements and keeps everything else the same. Consider the two cycle (ij) with $i < j$. We can put i in the spot of j by first applying $(i, i+1)$ then $(i+1, i+2)$ and so on until i is in the spot of j. This will leave all of the elements between i and j one lower and so j will be in the spot of $j-1$. This is because the last transposition in this process is $(j-1, j)$. So if we use transpositions to put j in the spot of i, by first applying $(j-2, j-1)$ then $(j-2, j-3)$ and so on, j will be in the spot of i and all of the elements in between will return to their original positions. So this sequence will yield an arbitrary 2-cycle, (ij). Every permutation can be written as a product of 2-cycles so the elements generate S_5.

26. Let $|G : H| = p$ and $|G : K| = q$. Suppose $p \neq q$ and $p < q$. So p and q divide $|G|$. So $|G|$ is of the form pqm. Because G is simple, $|G|$ divides $|G : H|! = p!$. So pqm divides $p!$ and therefore q divides $p!$. This is impossible because $p < q$. So $p = q$ and $|H| = |K|$.

28. Suppose G is a simple group that has a subgroup, H, that has index 4. By the embedding theorem $|G|$ is isomorphic to a subgroup of A_4, so $|G|$ divides $4!/2 = 12$ and is a multiple of 4. So $|G|$ has one of the following orders, $4, 8, 12$. If G has order 4 or 8 then the only prime divisor of $|G|$ is 2. So 1 is the only divisor of $|G|$ that is $1 \mod 2$. So G is not simple by Theorem 25.1.

If $|G| = |A_4|$ is 12, then G has at least one order 4 Sylow 2-subgroup. So a Sylow 2-subgroup has index 3 in G. But 12 does not divide $3!$. So G is not simple

Chapter 26

8. Let $a = \alpha\beta$ and $b = \alpha$. Note that $a = (1234)$ and $b = (12)(34)$ and $ab = (13)$. So $a^4 = b^2 = (ab)^2 = e$. So the set is isomorphic to a subgroup of D_4. The order of a is 4, as $a^2 \neq e$. So the set has at least 4 elements and b is not generated by a. So the set has a least 5 elements. The only subgroup of D_4 that has at least 5 elements has 8 elements and is D_4 itself.

26. Let $(a, b, c) = (0, 1, 1)$ and $(a, b, c) = (1, 0, 0)$. We get two matrices A and B satisfying the presentation of D_4: $A^2 = B^2 = I$, $(AB)^4 = I$. [One can further check that there are 8 elements.] The results follows.