Math 430 Homework 9 **Solution based on that of Liam Bench**

Chapter 29

2. The number of necklaces is the number of orbits of arrangements of beads under D_{16}. The number of ways to arrange 3 black beads and 13 white beads is $\binom{16}{3} = 560$.

1 identity fixes all 560 permutation.

15 non-trivial rotations fix no elements; if (j_1, j_2, j_3) are the positions of black beads then $(j_1 + k, j_2 + k, j_3 + k) \neq (j_1, j_2, j_3)$ modulo 16 for any k.

8 reflections along a line with 8 beads on each side of the line fix no elements, because there will be 2 black beads on one side, and 1 on the other.

8 reflections along a line passing two beads with 7 beads on each side fix 14 elements; for $j = 1, \ldots, 7$ choose $(j, j+8)$ to be (black,white) or (white,black) colors, then choose $(j+r)$ and it mirror image to be black.

The total number of designs is: $560 + 112)/32 = 21$.

4. The symmetric group for the benzene molecule is D_6, and identify the problem as coloring the vertexes of a regular hexagon with 3 colors. There are 3^6 different coloring without considering symmetry.

1 identity: fixes all 729 elements.

2 order-6 rotations: fix 3 elements; all vertexes will have the same color as vertex 1.

2 order-3 rotations: fix 9 elements; vertexes 1-2, 3-4, 5-6 have the same color pattern.

1 order-2 rotation: fixes 27 elements; vertexes 1-2-3 and 4-5-6 have the same color pattern.

3 reflections through a line with 3 vertexes on each side fix 27 elements.

3 reflections through a line passing 2 vertexes fix 81 elements.

The total number of different design is: $(1/12)(1 \cdot 3^6 + 1 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3 + 3 \cdot 3^4 + 3 \cdot 3^3) = 92$.

12. The symmetry group is $\mathbb{Z}_5$. (No reflection.) Identify the problem with coloring the vertexes of a pentagon with rotation symmetry. There are 3^5 coloring without symmetry.

1 identity fixes 243 elements.

4 elements of order 5 fix 3 elements with all vertexes having the same color.

Total number of designs $(1/5)(243 + 4 \cdot 3) = 51$.

Chapter 31

6. Let $\mathbf{0}$ be the zero vector. Note that $\mathrm{wt}(u) = d(\mathbf{0}, u)$ and for binary vectors, $\mathrm{wt}(u+v) = \mathrm{wt}(u-v)$. So $\mathrm{wt}(u) = d(\mathbf{0}, u) \leq d(u, v) + d(v, \mathbf{0}) = \mathrm{wt}(u - v) + \mathrm{wt}(v) = \mathrm{wt}(u + v) + \mathrm{wt}(v)$. So $\mathrm{wt}(u + v) \geq \mathrm{wt}(u) - \mathrm{wt}(v)$.

Suppose $\mathrm{wt}(u+v) = \mathrm{wt}(u) = \mathrm{wt}(v)$. Note that $\mathrm{wt}(u+v) = \sum(u_i + v_i \mod 2)$ where i indexes the ith component of the vector and $\mathrm{wt}(u) - \mathrm{wt}(v) = \sum u_i - \sum v_i$. If (u_i, v_i) is $(0, 0)$ or $(1, 1)$, this becomes 0 in both $\sum(u_i+v_i \mod 2)$ and $\sum u_i - \sum v_i$. So because $\sum(u_i+v_i \mod 2) = \sum u_i - \sum v_i$ and the only other number that is summed is 1, they have the same number of 1's. If $(u_i, v_i) = (1, 0)$ then $u_i + v_i \mod 2 = 1$ and $u_i - v_i = 1$. If $(u_i, v_i) = (0, 1)$ then $u_i + v_i \mod 2 = 1$ and $u_i - v_i = -1$. Suppose each side sums to j. So $j \cdot 1 = a \cdot 1 + b \cdot -1$ with $a + b = j$. So $a = j$ and $b = 0$. So there are no situations where $(u_i, v_i) = (0, 1)$.

If there are no situations where $(u_i, v_i) = (0, 1)$, then by the argument above $(u_i + v_i \mod 2) = u_i - v_i$ for each i. So their sums are equivalent.

10. This is a linear code with Hamming weight 4. So using Theorem 31.2 it can correct 1 error and can detect 3 or fewer errors.

14. Let $C = C_0 \cup C_1$ where C_0 (respectively, C_1) is the set of code words with last digit equal to 0 (respectively, 1). If C_1 is empty, we are done. If not, let $v \in C_1$, then the map $w \mapsto w + v$ will be a bijection on C exchanging C_0 and C_1. Thus, $|C_0| = |C_1|$.

16. We have the following table for the code:

00000	10011	01010	11001	00101	10110	01111	11100
10000	00011	11010	01001	10101	00110	11111	01100
01000	11011	00010	10001	01101	11110	00111	10100
00100	10111	01110	11101	00001	10010	01011	11000

So $11101 \to 110$ and $01100 \to 111$. If 11101 has one error it can be decoded to 110 or 111; so we cannot determine the intended word. If 01100 has one error it will be decoded to 111 correctly.

20. Each row is constructed by adding a vector of least weight. Here 10000 has least weight so it must have been added to the code words and we subtract it from each to obtain the code words. They are 00000, 10011, 01010, 11001, 00101, 10110, 0111, 11100.

24. Suppose the Hamming weight of a linear code is less than or equal to $2t$. So there is a code word, v, with j nonzero components where $1 \leq j \leq 2t$. Suppose went it is sent there are i errors that change i of the j nonzero components to 0s. If $j \leq t$ then v is received as the zero vector and the error cannot be corrected. If $j > t$ then it v is received as a vector t distance away from v and less than or equal to t distance away from the zero vector. So v is no longer the nearest-neighbor and cannot be decoded. Now suppose there are j errors that change all of the nonzero components to 0. So v is received as the zero vector and the error cannot be detected.

34. Let $C = C_o \cup C_e$ where C_o (respectively C_e) is the set of code words with odd (respectively, even) weights. If C_o is empty, we are done. If not, let $v \in C_o$, then the map $w \mapsto w + v$ will be a bijection on C exchanging C_o and C_e. Thus, $|C_o| = |C_e|$.