Additional topics

Definition Let G be a finite group, and S be a set of generators. The Cayley digraph $\mathrm{Cay}(S:G)$ has vertex set G and two vertex (g_1, g_2) is an arc if $g_1 s = g_2$ for some $s \in S$.

Examples

Definitions A Hamiltonian path (circuit) is a sequences of connected vertexes $(g_1, \ldots, g_k)$ such that every element of G appears once and only once (except for $g_k = g_0$).

Results

Theorem 30.1 If m, n are relatively prime, then $\mathrm{Cay}(\{(1,0),(0,1)\}, Z_m \oplus Z_n)$ has no Hamiltonian circuit.

Proof. If (a,b) go to $(a,b+1)$, then $(a-1,b+1)$ will go to $(a-1,b+2)$....

Theorem 30.2 If m divides n, then $\mathrm{Cay}(\{(1,0),(0,1)\}, Z_m \oplus Z_n)$ has a Hamiltonian circuit.

Proof.

Theorem 30.3 *If G is Abelian, then $\mathrm{Cay}(S,G)$ has a Hamiltonian path.*

Proof.

Remarks Find Eulerian circuits.

My interest Determine the diameter of a given Cayley graph of S_n.

Find the shortest path from a permutation to the identity.

This is the number of steps needed in a specific sorting algorithm.

Definition An isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function that preserve distance.

Definition Let $F \subseteq \mathbb{R}^n$. The symmetry group of F is the set of all isometries of $\mathbb{R}^n$ that maps $\mathbb{F}$ to itself using function composition as the group operation.

Remark Let G be a the group of isometries of a line segment. It has order 2, 4 or infinity depends G is in M_1, M_2, or M_k for $k \geq 3$.

Fact Every isometry on $\mathbb{R}^2$ is one of the four types: rotation, translation, reflection, and glide-reflection.

Classification theorem

Theorem 27.1 The only finite plane symmetry group (in M_2) are $\mathbb{Z}_n$ and D_n.

Idea of proof. Non-trivial translation is not allowed.

If there are two reflections, then the line of symmetry intersect.

The group will be rotation about this point, and reflections about some lines of symmetry passing through this point. □

Theorem 27.2 Finite groups of rotation in $\mathbb{R}^3$ are $\mathbb{Z}_n, D_n, A_4, S_4, A_5$.

Remark Some of these groups are the isometry groups of the Plutonic solids, regular polyhedrons in $\mathbb{R}^3$.

There are 7 types Frieze groups for decorative designs.

- $\mathbb{Z} = \langle x \rangle$ with $x =$ translation.
- $\mathbb{Z} = \langle x \rangle$ with $x =$ glide-reflection.
- $D_\infty = \langle x, y \rangle$ with $x =$ translation, $y =$ vertical reflection.
- $D_\infty = \langle x, y \rangle$ with $x =$ glide-reflection, $y = 180^\circ$-reflection.
- $D_\infty = \langle x, y \rangle$ with $x =$ glide-reflection, $y = 180^\circ$-reflection.
- $\mathbb{Z} \oplus \mathbb{Z}_2 = \{x^n y^m : n \in \mathbb{Z}, m \in \mathbb{Z}_2\}$ with
  $x =$ translation, $y =$ horizontal reflection.
- $\mathbb{Z} \oplus \mathbb{Z}_2 = \{x^n y^m z^k : n \in \mathbb{Z}, m \in \mathbb{Z}_2, z \in \mathbb{Z}_2\}$ with
  $x =$ translation, $y =$ horizontal reflection, $z =$ vertical reflection.

Further remarks

- There are 17 additional types of discrete plane symmetry groups.
- There are 230 three-dimensional crystallographic groups (space groups).
- The are 4783 four-dimensional symmetry groups for infinitely repeating patterns.
- Hilbert (1900) conjectured and Bieberbach (1910) proved that there are always finitely many crytallographic groups in n dimensions.
- The results are useful in the study of X-ray diffraction concerning the molecular structure, and also DNA structure.