Amidakuji, sorting algorithms, & permutations:

From recreational mathematics to research mathematics

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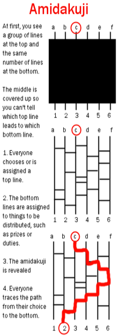
Amidakuji/Ghost Leg Drawing



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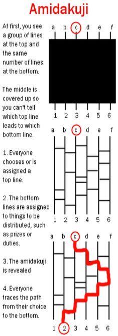

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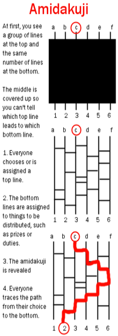

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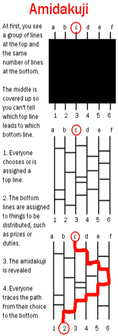

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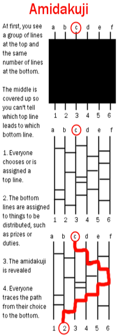

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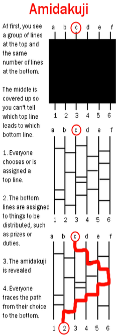

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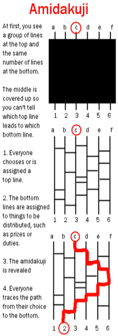

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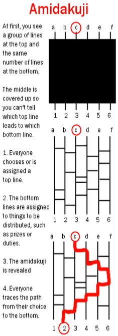

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- Why do we always get an one-one correspondence (bijection)?
- Can we get all possible job assignments?
- What is the minimum number of horizontal segments needed for a given job assignment?

Answer of Question 1

George Polya (1887-1985)

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- What if there is no horizontal line segment?
- What if there is one horizontal line segment?
- An easy induction argument!

Bubble sort

- Record the job assignment as a permutation (a seat reassignment), say,

$$\sigma = [5,3,1,2,4] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}.$$

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- **Key idea** Determine the number $\iota(\sigma)$ of inversions of σ.

Example For $\sigma = [5, 3, 1, 2, 4]$, total number of inversions is:

$$\iota(5) + \iota(4) + \iota(3) + \iota(2) + \iota(1) = 4 + 0 + 2 + 0 + 0 = 6,$$

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$$(n-1)+\cdots+1 = n(n-1)/2 \text{ steps.}$$

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Example. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 9 & 6 & 7 & 1 & 8 & 2 \end{pmatrix} = (1,3,5,6,7)(2,4,9)(8).$

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Then $\sigma = (1,7)(1,6)(1,5)(1,3)(2,9)(2,4).$

Some open problems

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- One can write a computer program to determine r^*.

Partial results of the general problem

We have the following list for $r^*(n,m)$ for S_n and $(i, i+\ell)$ with $\ell \leq m$,

$n \backslash m$	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	3	2									
4	6	4	3								
5	10	5	5	4							
6	15	[7]	6	6	5						
7	21	[10]	8	7	7	6					
8	28	[14]	[10]	9	8	8	7				
9	36	[16]	[11]	10	10	9	9	8			
10	45	[19]	[14]	[12]	11	11	10	10	9		
11	55	[23]	[16]	[14]	13	12	12	11	11	10	
12	66	29^*	20^*	17^*	16^*	14	13	13	12	12	11

where the entries marked by brackets are obtained by computer programming.

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- Quantum computing.

  Decompose certain quantum gates into simple quantum gates (CNOT gates).

Reference:

Zejun Huang, Chi-Kwong Li, Sharon H. Li, Nung-Sing Sze, Factorization of permutations, preprint, arXiv:1303.3776.

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Thank you for your attention!

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- If we use $(1,2)$, the forward and backward long cycles $(1,\ldots,n)$ and $(1,n,n-1,\ldots,2)$, then r^* is given by:

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Conjecture $r^* = \binom{n}{2}$ for $n \geq 4$.