

Fibonacci Numbers in Daily Life

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Abstract

Fibonacci sequence was first introduced by an Italian mathematician Fibonacci in his book Liber Abaci. People named this sequence with Fibonacci's name because it is closely connected with nature, science and real life. Therefore Fibonacci sequence is widely used in many fields and well-worth people exploring. In this paper, we will introduce two properties of Fibonacci hidden in a game, and further application of Fibonacci number mainly in botany and composition of pictures.

Key Words: Fibonacci sequence; golden ratio; application.

1 Introduction

Fibonacci sequence was first introduced by an outstanding Italian mathematician Fibonacci (Leonardo Pisano.1175-1250) in his book Liber Abaci (1202). For he was born in the commercial center of Pisano, he was also called Leonardo of Pisano^[1]. When Fibonacci was a little child, he traveled with his father who worked as a businessman. He also traveled to other countries like Egypt, Greek and Syria to study mathematic knowledge from the east when he grew up.

Fibonacci sequence begins with $a_0 = 0, a_1 = 1$, and each subsequent number is the sum of the previous two numbers, so the recursion formula of Fibonacci sequence is $a_{n+1} = a_n + a_{n-1}$. People thought the sequence contains the secret of nature, so they named it with the mathematician's name Fibonacci. In fact, Fibonacci sequence closely connects with nature, science and real life. Therefore it is widely used in many fields and well-worth people exploring.

2 Game about Fibonacci number

2.1 Game

Cut a 8×8 square into four parts(as figure 1 shows), and rearrange the four parts into a new 5×13 rectangle as figure 2 shows.

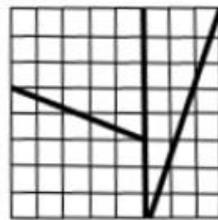


Figure 1 Original square

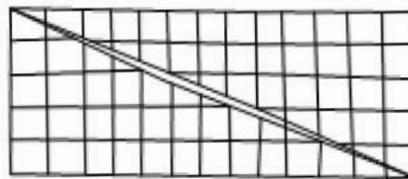


Figure 2 Rearrange into a new rectangle^[2]

When we calculate the area of figure 1, we can easily get that the area of the square equals to 64. However, the area of the new rectangle equals to 65. During this process, we have not abandoned or added any piece of paper into the new rectangle, so the area of the original square and the rectangle should equal to each other. In order to figure out the reason for the area change and the hidden mathematical principle behind, we picked this game as our topic.

2.2 Mathematic principle behind

We observed the square carefully and found out that the length of the sides of the square and the rectangle are all Fibonacci numbers. We have

$$n = 6$$

$$a_n = a_6 = 8, a_{n-1} = a_5 = 5 \text{ and } a_{n+1} = a_7 = 13.$$

In fact, Fibonacci sequence has the following property:

$$a_{n+1}a_{n-1} = a_n^2 + (-1)^n \quad (1)$$

In our game, the product of $a_{n+1}a_{n-1}$ actually represents the area of the rectangle

$S_{\text{rectangle}}$, and a_n^2 represents the area of the square S_{square} .

$$S_{\text{rectangle}} = a_7a_5 = 5 \times 13 = 65,$$

$$S_{\text{square}} + (-1)^6 = a_6^2 + (-1)^6 = 64 + (-1)^6 = 65.$$

If we observe the new rectangle carefully, we will find there is a gap on the rectangle. So the area of the original square has never changed. The area change only caused by adding extra part into the area of the rectangle.

We use the mathematical induction method to prove this property:

Proof: Step 1 Let $n = 1$, $a_0 = 0, a_1 = 1, a_1 = 1$.

Then equality (1) holds.

Step 2 Suppose it is true for $n=k$. Then we get the equality

$$a_{k-1}a_{k+1} = a_k^2 + (-1)^k \quad (2)$$

Step 3 Now we show it is true for $n=k+1$.

According to the definition of the Fibonacci sequence, we have the following recursion formula:

$$a_{k+1} = a_k + a_{k-1};$$

$$a_{k+2} = a_{k+1} + a_k = 2a_k + a_{k-1}$$

$$\text{Then we get } a_{k-1}(a_k + a_{k-1}) = a_k^2 + (-1)^k$$

$$a_k^2 - a_{k-1}a_k - a_{k-1}^2 = (-1)^{k+1}$$

$$a_k(2a_k + a_{k-1}) = (a_k + a_{k-1})^2 + (-1)^{k+1}$$

Thus

$$a_k a_{k+2} = a_{k+1}^2 + (-1)^{k+1} \quad (2)$$

It is true for $n=k+1$.

Thus property(1) is proved.

2.3 Another situation

When we reshape the four parts of the original square in another way as figure 3 shows, we get a new polygon. The intersection in the new polygon changes its area into 63. We can also use another property of Fibonacci sequence to explain why the change happens.

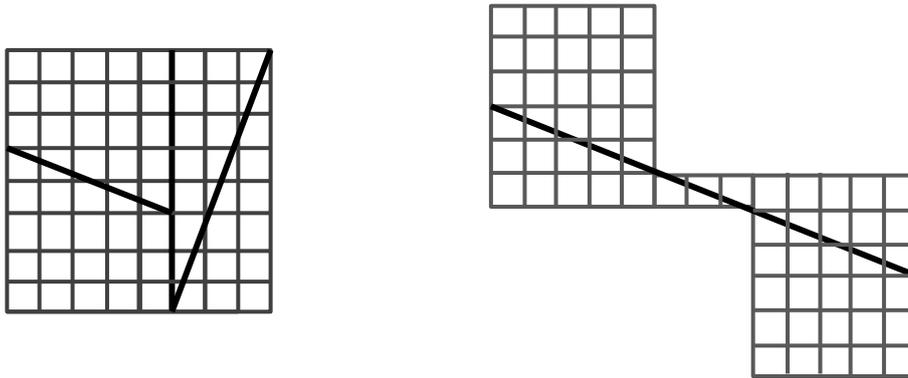


Figure 3 Rearrange four parts into a new polygon

The Fibonacci sequence also has the following property ^[3]:

$$4a_{n-1}a_{n-2} + a_{n-2}a_{n-4} = a_n^2 + (-1)^{n-1}$$

Proof For we have (2) $a_n a_{n+2} = a_{n+1}^2 + (-1)^{n+1}$

$$\begin{aligned} 4a_{n-1}a_{n-2} + a_{n-2}a_{n-4} &= 4a_{n-1}a_{n-2} + a_{n-2}(2a_{n-2} - a_{n-1}) \\ &= a_{n-2}(3a_{n-1} + 2a_{n-2}) \\ &= a_{n-2}(a_n + a_{n-2} + 2a_{n-1}) \\ &= a_{n-1}^2 + (-1)^{n-1} + a_{n-2}^2 + 2a_{n-2}a_{n-1} \\ &= a_n^2 + (-1)^{n-1} \end{aligned}$$

Thus the property is proved.

When we calculate the area of the new polygon, we actually do not involve the intersection part. So the area of the new polygon we get is smaller than the original square.

2.4 Perfect situation

After we had researched two situations above, it is reasonable to think about the question that how can we cut the original square to get a perfect rectangle. Suppose the length of the sides are x and y as figure 4 shows, and we can get a new rectangle without any gap or intersection.

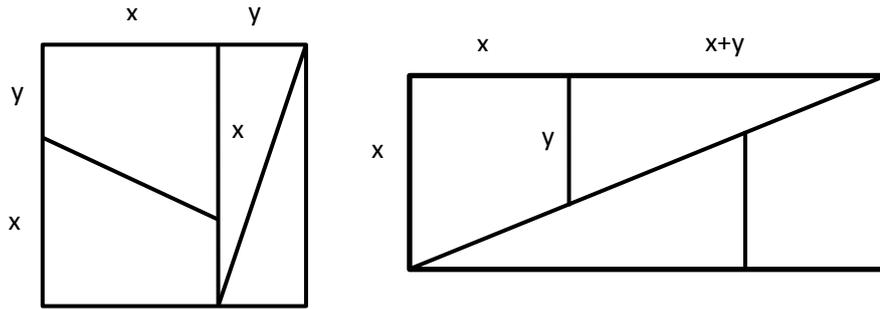


Figure 4 The situation we can get a perfect rectangle

Suppose the area of the original square is S_{square} and the area of rectangle is $S_{rectangle}$. Then we have

$$S_{square} = (x + y)^2$$

$$S_{rectangle} = (2x + y)x$$

Let

$$S_{square} = S_{rectangle}$$

Then we have

$$(x + y)^2 = (2x + y)x$$

i.e.

$$\left(\frac{x}{y}\right)^2 - \left(\frac{x}{y}\right) - 1 = 0$$

Thus we get the solution

$$\frac{x}{y} = \frac{1 \pm \sqrt{5}}{2}$$

Because x and y are the length of the sides, so we only take the positive one. It is easy to realize that

$$\frac{y}{x} = \frac{2}{1 + \sqrt{5}} \approx 0.618$$

In fact, if y equals to a_{n-2} and x equals to a_{n-1} , the proportion of x for y represents the proportion of a_{n-1} for a_{n-2} . When n tends to infinity, the ratio tends to be 0.618 (golden ratio). So the perfect situation above happened when x equals to a_{n-1} , y equals to a_{n-2} and n tends to be infinity.

2.5 Fibonacci numbers and golden ratio

From the result above, we find that Fibonacci numbers are close connected with golden ratio. Therefore in the following part, we will apply the continued-fraction approximation to show the connection between them.

The golden ratio is very useful in Aesthetics and the Optimization, for example, the golden rectangle and the golden ratio method. But in some cases, the golden ratio method can't be used directly, for example, when we deal with the Optimization problem whose domain is a set of discrete points, which means the total length of the

domain can be computed as an integer, but the domain is discrete. So find some fractions close to the golden ratio would be the solution for the problem mentioned above.

The following listed is some conclusion we get by analyzing the problem.

Property 1:

Obviously, if we decrease the error of the approximation, the denominator of the fraction will be increased. But for a concrete discrete Optimization problem, its domain length is a concrete integer, and only when the denominator of the fraction equals that integer can that fraction be used to solve the problem.

Corollary 1:

So we need fractions with different denominators to solve the discrete Optimization problem with different domain length.

Property 2:

However, when the denominator of the fraction less than the domain length of the problem, we can also find a way to use that fraction to solve the problem. The idea is simply by dividing the domain of the problem into several parts and the length of each part is the denominator of the fraction except the last part whose length is less than it. Then we can use the fraction to search the optimum part by part, but for the last part we need use dichotomization to find the optimum of this part. Finally, by simply comparing all the optimums and choosing the best one of them, we can get the problem solved.

Corollary 2:

The best fraction approximating the golden ratio varies with the change of the domain length of the problem.

Definition 1:

We define the best fraction approximating the golden ratio as $G[n]$, when we limit the denominator of the fraction less than an integer n . So the denominator of $G[n]$ is less than n , and among all fractions with denominator less than n , $G[n]$ is the best one to approximate the golden ratio.

Till now we have get all the preliminaries prepared to show the connection between the Fibonacci sequence and the golden ratio.

What we want to show below is an equation:

$$G[n] = \frac{F[k-1]}{F[k]} \text{ when } F[k-1] < n \leq F[k]$$

Where the sequence denoted by $\{F[k]\}$ is the Fibonacci sequence.

Limited by space, we put the detailed proof of the above equation in the appendix.

3 Further applications

3.1 Application in botany



Figure 5 The Fibonacci spiral appeared in some kind of aloe



Figure 6 Flowers whose numbers of petals are Fibonacci numbers

Considering the number of the petals of the flowers, some of them follow Fibonacci numbers. One possible reason for why this phenomenon happens is that they try to decrease the overlapped area to get more sunlight.

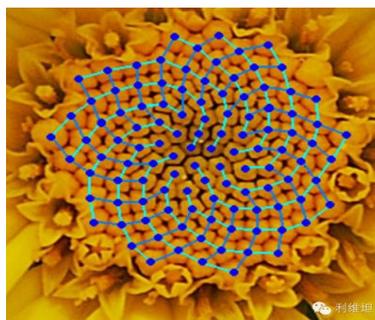


Figure 7 The hidden Fibonacci spiral in sunflower

According to figure 7, when we count the number of its spirals, we will find there are 13 counterclockwise spirals and 21 clockwise spirals. These two numbers are both Fibonacci numbers. One possible reason for the Fibonacci spiral appears in the sunflower is to fully use the space.

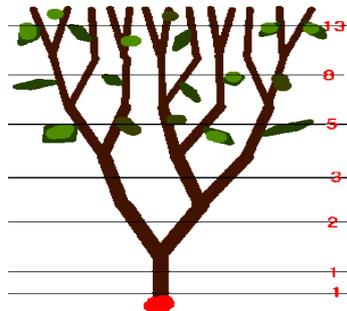


Figure 8 The number of branches follows the Fibonacci numbers

Botanists also find the numbers of branches of trees are always Fibonacci number. They find that after a certain period of time, each old branch of a tree will get a new one and need one more period of time to turn to an old one. If the tree has only one branch at very beginning, after a year it will have two branches and in next circle, it has three branches. Every year the total branches of the tree composed a Fibonacci sequence. One possible reason for it is that every new branch needs one year to get mature.

Since Fibonacci numbers shows in some of the flower petal numbers and the arrangements of the inflorescence, many plants do not show any Fibonacci number in their arrangements ^[4]. In fact, scientists have not figured out the pattern accurately the plants arrange their leaves and flowers. We can only assume that plants apply Fibonacci numbers and the golden ratio in their growth to make their space utilizations more effectively. However, it is also possible that plants do not understand Fibonacci sequence at all, and maybe they just grow to the shape they like.

3.2 Application in composition

3.2.1 Fibonacci spiral

One of the most common applications of Fibonacci sequence in the composition is Fibonacci spiral. As figure 5 shows, the sides of the square are all different Fibonacci numbers.

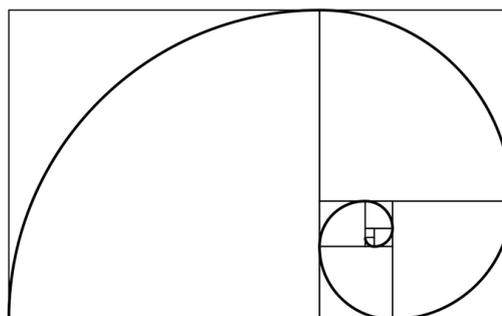


Figure 5 Fibonacci spiral

The following pictures show the hidden Fibonacci spirals in composition.

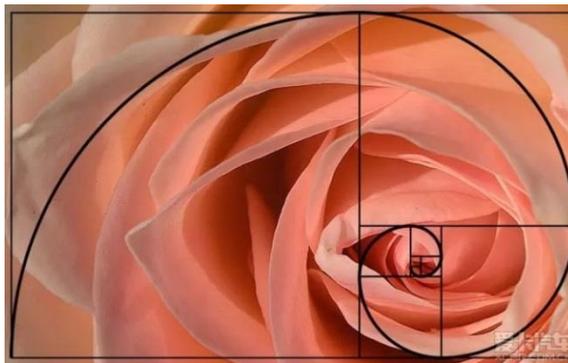


Figure 6 Picture of rose

The photographer use the spiral into composition to highlight the rose in figure 6. Figure 6 shows the hidden Fibonacci spiral in the picture.

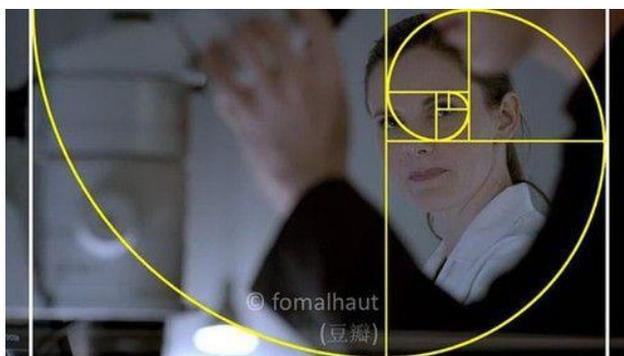


Figure 7 Stage photo of Sherlock

Figure 7 is a stage photo picked from BBC's show Sherlock. The starting point of Fibonacci spiral in Figure 7 highlights the girl's eye, and the arm of the man help form the spiral. The application of Fibonacci spiral can make the picture looked more harmonious ^[5].

3.2.2 The Golden Triangle

The triangle ABC and triangle CBX are both isosceles triangles and similar to each other. The waist and the bottom edge length ratio for 1.618. Figure 8 shows how a golden triangle looks like.

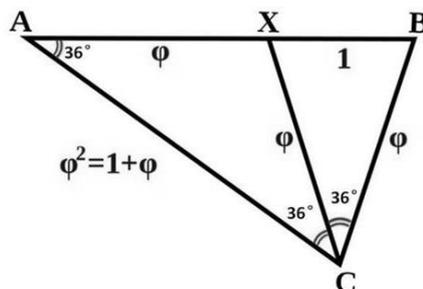


Figure 8 The Golden Triangle

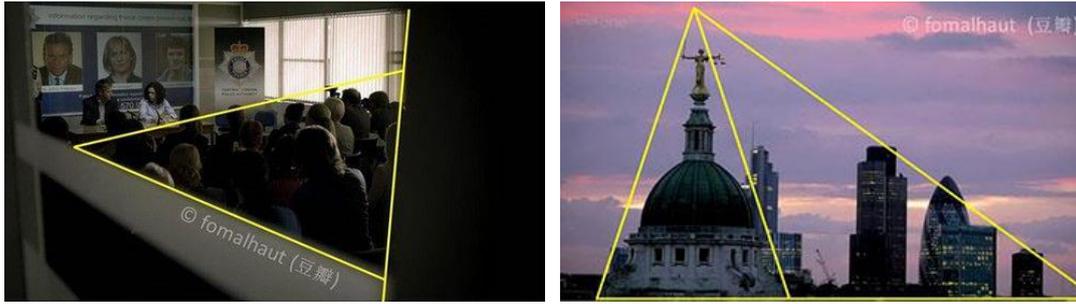


Figure 9 Golden triangle used in stage photos

Figure 9 shows the hidden golden triangle being applied in two stage photos. The application of the golden triangle also brings the sense of beauty to the pictures.

3.3 Application in bees

The breed form of bees is very special, for the male bees are all hatched from unfertilized eggs and the female ones are all hatched from fertilized one. Hence the male bees are only the fathers of the female ones, and male bees also have no father at all. When scientists observed the family tree of the bees, they found that the number G_n of the n generation of the first male bee equals to the a_n in Fibonacci sequence.

Figure 10 shows the family tree of a male bee, the symbol ♀ represents female bees and the symbol ♂ represents the male bees.

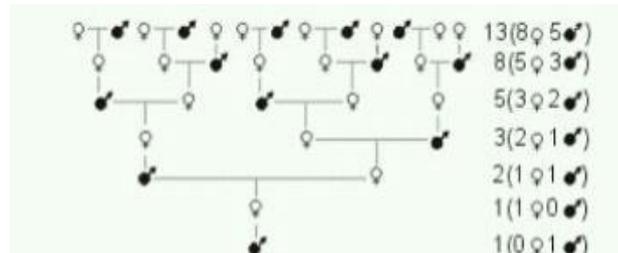


Figure 10 The family tree of a male bee ^[6]

Reference

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Appendix

The proof for the equation:

$$G[n] = \frac{F[k-1]}{F[k]} \text{ when } F[k-1] < n \leq F[k]$$

Where the sequence denoted by $\{F[k]\}$ is the Fibonacci sequence.

If we denote the golden ratio by a label x , then according to the definition of the golden ratio we get the label x satisfying the following equation.

$$\frac{1-x}{x} = x$$

Where the variable denoted by x belong to the interval $[0, 1]$.

Now we use elementary mathematics to compute $G[n]$.

By solving the equation above we get

$$x \approx 0.61803398874989484820458683436564$$

$$x = \frac{1}{\frac{1}{x}} = \frac{1}{\text{floor}\left(\frac{1}{x}\right) + \left[\frac{1}{x} - \text{floor}\left(\frac{1}{x}\right)\right]}$$

$$0 \leq \frac{1}{x} - \text{floor}\left(\frac{1}{x}\right) < 1$$

The label $\text{floor}(x)$ represents the greatest integer less or equal to the number x .

Obviously, $\text{floor}\left(\frac{1}{x}\right) > 1$, we can neglect $\frac{1}{x} - \text{floor}\left(\frac{1}{x}\right)$ and get a approximation of x , which mean:

$$x \approx \frac{1}{\text{floor}\left(\frac{1}{x}\right)}$$

Next we will prove:

$$G\left[\text{floor}\left(\frac{1}{x}\right)\right] = \frac{1}{\text{floor}\left(\frac{1}{x}\right)}$$

Proof:

If $G\left[\text{floor}\left(\frac{1}{x}\right)\right] = \frac{q}{p}$, then $1 \leq q \leq p \leq \text{floor}\left(\frac{1}{x}\right) \leq \frac{1}{x}$ and we need $\frac{q}{p} - x$ reach its minimum.

$$\frac{q}{p} - x = \frac{q - p \cdot x}{p}$$

Based on the given conditions, we get $p \cdot x \leq 1$, so by letting the numerator reaching its minimum and letting the denominator reaching its maximum we get:

$$q = 1 \quad , \quad p = \text{floor}\left(\frac{1}{x}\right)$$

then we proved that:

$$G\left[\text{floor}\left(\frac{1}{x}\right)\right] = \frac{1}{\text{floor}\left(\frac{1}{x}\right)}$$

Obviously,

$$G[1] = 1$$

To find the next fraction with a bigger denominator and higher accuracy, we can simply use the method above to find a fraction to approximate the number——

$$\frac{1}{x} - \text{floor}\left(\frac{1}{x}\right) .$$

$$\left[\frac{1}{x} - \text{floor}\left(\frac{1}{x}\right)\right] = \frac{1}{x} - 1 = \frac{1-x}{x} = x$$

So the method to compute $G[1]$ can be iteratively used to compute $G[n]$ and then we can get a sequence which is a subset of $\{G[n]\}$ (we neglect the detailed proof for this assertion here, but it can be proved by using the mathematical induction) . What's more, if we denote the sequence by $\{x[n]\}$, then we can get a recursive relation:

$$x[n] = \frac{1}{1+x[n-1]}, x[0] = G[1] = 1.$$

It would be easy to show the formula—— $x[n] = \frac{F[n]}{F[n+1]}$.

We neglect the proof of the set $\{G[n]\}$ is equal to the set $\{x[n]\}$, because the proof need to use many properties about the continued-fraction approximation theory. If you feel interested in it you can consult Wikipedia's entry on continued-fraction. Actually the continued-fraction approximation method can also be applied to solve the problem exists in the twelve-tone equal temperament.

At last, by simply applying the theory about the Fixed-Point Iteration we can verify that $\{x[n]\}$ converges to the golden ratio x .

Note

In fact, Fibonacci number is very interesting and we can see it in many objects. Especially in nature, such as the sorting of the petals, and leaves, they are consistent with Fibonacci numbers. Fibonacci also exist in insects such as the honeycomb.

--By Huixia Chen

Through this class and by doing the group assignment of it, I learned that life is the source and impetus of the development of mathematics. So keeping a vigorous eye on the problems appeared in real life and solving them by using mathematical idea will certainly give you some ideas to develop new mathematics.

--By Wangchun Peng

In this course, we actually learn more about 'daily life' than 'mathematic'. We did not do the endless calculation, did not prove endless theorems and did not solve endless equations. What we did was finding, learning and improving. During this whole week, we work hard to find the mathematics hidden in our daily lives, learn from other students from our class, and never stop trying to improve ourselves.

From the team work I learnt how to communicate with others, how to work efficiently and how to be patient. From the presentation I learnt how to consider problems in different ways, how to give a better presentation, and to respect the labors of others.

I used to feel cool when I understood the math theory that others could not understand at all. But I gradually realized that people do not care what you can do only in mathematic field, but they care about what you can do to make the world better. There are several name of mathematicians in my text book my mum has never heard about. But people always remember the ones who devoted themselves to the society even though the ones who did not get high education. I do not expect to let people remember me, but I want to make something in the world better.

When I sat down and looked back the whole week, I got the thinking above. I hope I can keep the idea in my mind: never stop trying figure out things, never stop improving ourselves and never forget to take others' shoes.

--By Yining Lin

I think the idea of the course Mathematics in Daily Life is to provide more learning methods and thinking to my math study. This course also helps me solving the problems in our daily lives and stimulates my enthusiasm for English

--By Ya Liu

Work division

Yining Lin : presentation; slide show ; paper writing

Wangchun Peng: presentation; slides show; paper writing

Huixia Chen: presentation; paper writing; data collection

Ya Liu: topic choose; slides show (absent from presentation and low join rate in paper writing)