

# Urban Services: Eulerian Graph

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# Objectives of presentation expected learning outcome

Discuss how to use **graph theory** to **model** and **solve** problems in **Management science**, also known as **operations research**.

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Eulerian Graph

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Urban Services

Seven bridge  
problem

Solution of  
Euler

Leonhard Euler

Euler's  
Theorem

# Some sample problems

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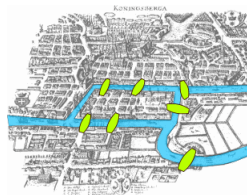
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- **(Chinese) postman problem** - start from one point (post office), deliver mail to each sidewalk, and return to the starting point without repeating too many sidewalks.

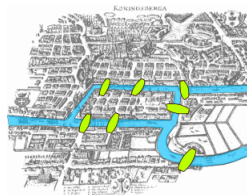
## Seven bridge problem

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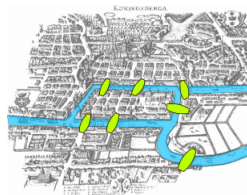
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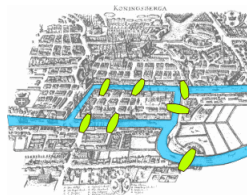
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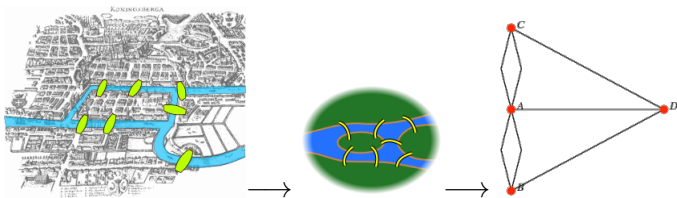
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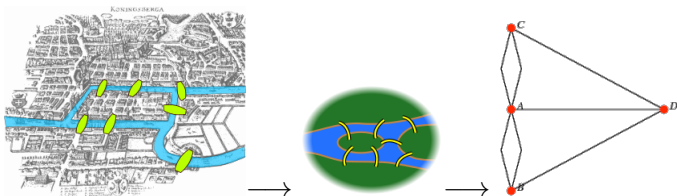
- The islands could not be reached by any route other than the bridges.
- Every bridge must have been crossed completely every time.

## The solution of Euler, 1736



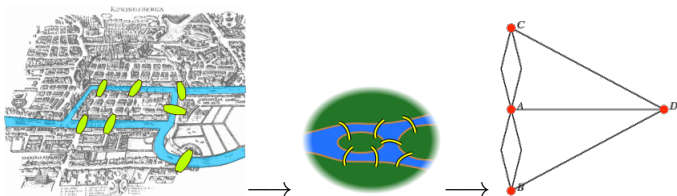
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- We need a **circuit** which covers all the edges once and exactly once. Such a circuit is called an **Euler Circuit**.
- If a graph has an Euler circuit, then (1) the graph has to be **connected**, and (2) every vertex has an **even** degrees.

## Leonhard Euler, 1707-1783

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- He continued to do mathematics and produced interesting results even after he became totally blind later in his life.
- People commented that: Euler could calculate effortlessly, “just as men breathe, as eagles sustain themselves in the air”.



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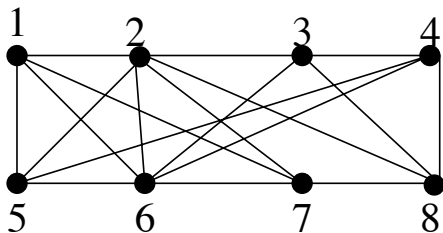
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- Combine  $C$  with the Euler circuits to get an Euler circuit in the original graph.

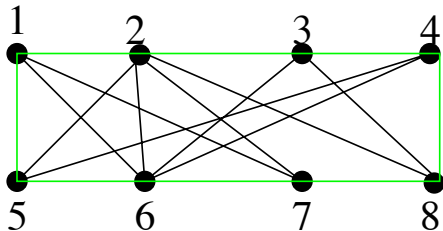
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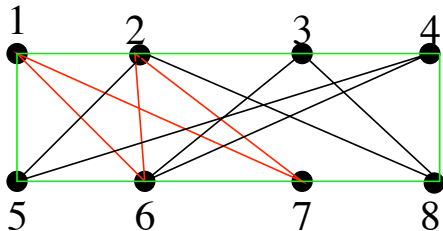
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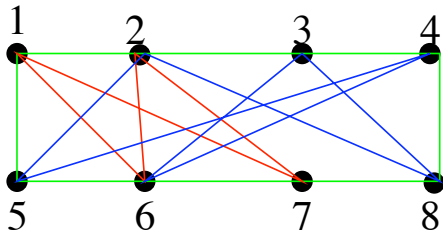
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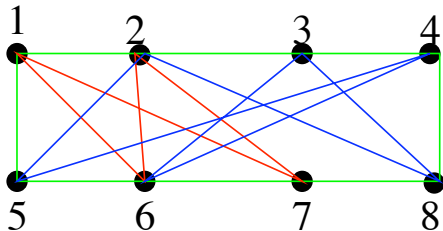


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The end!