The Counterfeit Coin Problem

Chi-Kwong Li
Ferguson Professor of Mathematics
The College of William and Mary

Outline

Outline

- Describe a logic game on finding a counterfeit coin.

Outline

- Describe a logic game on finding a counterfeit coin.
- Explain the mathematics behind.

Outline

- Describe a logic game on finding a counterfeit coin.
- Explain the mathematics behind.
- Extend the method to other situations.

Outline

- Describe a logic game on finding a counterfeit coin.
- Explain the mathematics behind.
- Extend the method to other situations.
- Explore its connection to other problems.

A Simple Problem

Problem

Suppose 27 coins are given. One of them is fake and is lighter. Find the minimum number of weighting using a balance with two pans to find the fake coin.

A Simple Problem

Problem

Suppose 27 coins are given. One of them is fake and is lighter. Find the minimum number of weighting using a balance with two pans to find the fake coin.

A Simpler Problem What about 9 coins?

A Simple Problem

Problem

Suppose 27 coins are given. One of them is fake and is lighter. Find the minimum number of weighting using a balance with two pans to find the fake coin.

A Simpler Problem What about 9 coins?

An Even Simpler Problem What about 3 coins?

A Simple Problem

Problem

Suppose 27 coins are given. One of them is fake and is lighter. Find the minimum number of weighting using a balance with two pans to find the fake coin.

A Simpler Problem What about 9 coins?

An Even Simpler Problem What about 3 coins?

Solution

If there are 3^m coins, we need only m weightings.

A Simple Problem

Problem

Suppose 27 coins are given. One of them is fake and is lighter. Find the minimum number of weighting using a balance with two pans to find the fake coin.

A Simpler Problem What about 9 coins?

An Even Simpler Problem What about 3 coins?

Solution

If there are 3^m coins, we need only m weightings.

What if there are 8 coins?

Extension of the idea

Table 1

$k:$	$1-3$	$4-9$	$10-27$	$28-81$	$82-243$
$m:$	1	2	3	4	5

Extension of the idea

Table 1

$k:$	$1-3$	$4-9$	$10-27$	$28-81$	$82-243$
$m:$	1	2	3	4	5

Extension of the idea

Table 1

$k:$	$1-3$	$4-9$	$10-27$	$28-81$	$82-243$
$m:$	1	2	3	4	5

More generally, if there are k coins with $3^{m-1} < k \leq 3^m$, then we need only m weightings.

Extension of the idea

Table 1

k :	$1-3$	$4-9$	$10-27$	$28-81$	$82-243$
m :	1	2	3	4	5

More generally, if there are k coins with $3^{m-1} < k \leq 3^m$, then we need only m weightings.

The general formula $\lceil \log_3(k) \rceil$.

Extension of the idea

Table 1

$k:$	$1-3$	$4-9$	$10-27$	$28-81$	$82-243$
$m:$	1	2	3	4	5

More generally, if there are k coins with $3^{m-1} < k \leq 3^m$, then we need only m weightings.

The general formula $\lceil \log_3(k) \rceil$.

Deeper ideas Tree diagram/graph. Divide and conquer algorithm.

Extension of the idea

Table 1

$k:$	$1-3$	$4-9$	$10-27$	$28-81$	$82-243$
$m:$	1	2	3	4	5

More generally, if there are k coins with $3^{m-1} < k \leq 3^m$, then we need only m weightings.

The general formula $\lceil \log_3(k) \rceil$.

Deeper ideas Tree diagram/graph. Divide and conquer algorithm.

Generalization Suppose one has a three pan balance. Then one can find the fake coin out of k coins by m weightings with $4^{m-1} < k \leq 4^m$.

Extension of the idea

Table 1

$k:$	$1-3$	$4-9$	$10-27$	$28-81$	$82-243$
$m:$	1	2	3	4	5

More generally, if there are k coins with $3^{m-1} < k \leq 3^m$, then we need only m weightings.

The general formula $\lceil \log_3(k) \rceil$.

Deeper ideas Tree diagram/graph. Divide and conquer algorithm.

Generalization Suppose one has a three pan balance. Then one can find the fake coin out of k coins by m weightings with $4^{m-1} < k \leq 4^m$.

If there is a p pan balance then ....

A More Difficult Problem

Suppose 12 coins are given such that one of them has a different weight. Use three weightings to find the different coin, and determine whether it is heavier or lighter.

A More Difficult Problem

Suppose 12 coins are given such that one of them has a different weight. Use three weightings to find the different coin, and determine whether it is heavier or lighter.

Solution

Compare $\{A1, A2, A3, A4\}$ and $\{B1, B2, B3, B4\}$.

- If equal, compare $\{A1, A2, A3\}$ and $\{C1, C2, C3\}$.

A More Difficult Problem

Suppose 12 coins are given such that one of them has a different weight. Use three weightings to find the different coin, and determine whether it is heavier or lighter.

Solution

Compare $\{A1, A2, A3, A4\}$ and $\{B1, B2, B3, B4\}$.

- If equal, compare $\{A1, A2, A3\}$ and $\{C1, C2, C3\}$.
- If $\{A1, A2, A3, A4\} > \{B1, B2, B3, B4\}$, compare $\{A1, A2, B1\}$ and $\{A3, B2, C1\}$.

A More Difficult Problem

Suppose 12 coins are given such that one of them has a different weight. Use three weightings to find the different coin, and determine whether it is heavier or lighter.

Solution

Compare $\{A1, A2, A3, A4\}$ and $\{B1, B2, B3, B4\}$.

- If equal, compare $\{A1, A2, A3\}$ and $\{C1, C2, C3\}$.
- If $\{A1, A2, A3, A4\} > \{B1, B2, B3, B4\}$, compare $\{A1, A2, B1\}$ and $\{A3, B2, C1\}$.
- If $\{A1, A2, B1\} > \{A3, B2, C1\}$, then compare $\{A1\}$ and $\{A2\}$.

A More Difficult Problem

Suppose 12 coins are given such that one of them has a different weight. Use three weightings to find the different coin, and determine whether it is heavier or lighter.

Solution

Compare $\{A1, A2, A3, A4\}$ and $\{B1, B2, B3, B4\}$.

- If equal, compare $\{A1, A2, A3\}$ and $\{C1, C2, C3\}$.
- If $\{A1, A2, A3, A4\} > \{B1, B2, B3, B4\}$, compare $\{A1, A2, B1\}$ and $\{A3, B2, C1\}$.
- If $\{A1, A2, B1\} > \{A3, B2, C1\}$, then compare $\{A1\}$ and $\{A2\}$.
- If $\{A1, A2, B1\} < \{A3, B2, C1\}$, then compare $\{C1\}$ and $\{A3\}$.

A More Difficult Problem

Suppose 12 coins are given such that one of them has a different weight. Use three weightings to find the different coin, and determine whether it is heavier or lighter.

Solution

Compare $\{A1, A2, A3, A4\}$ and $\{B1, B2, B3, B4\}$.

- If equal, compare $\{A1, A2, A3\}$ and $\{C1, C2, C3\}$.
- If $\{A1, A2, A3, A4\} > \{B1, B2, B3, B4\}$, compare $\{A1, A2, B1\}$ and $\{A3, B2, C1\}$.
- If $\{A1, A2, B1\} > \{A3, B2, C1\}$, then compare $\{A1\}$ and $\{A2\}$.
- If $\{A1, A2, B1\} < \{A3, B2, C1\}$, then compare $\{C1\}$ and $\{A3\}$.
- If $\{A1, A2, B1\} = \{A3, B2, C1\}$, then compare $\{B3\}$ and $\{B4\}$.

A More Difficult Problem

Suppose 12 coins are given such that one of them has a different weight. Use three weightings to find the different coin, and determine whether it is heavier or lighter.

Solution

Compare $\{A1, A2, A3, A4\}$ and $\{B1, B2, B3, B4\}$.

- If equal, compare $\{A1, A2, A3\}$ and $\{C1, C2, C3\}$.
- If $\{A1, A2, A3, A4\} > \{B1, B2, B3, B4\}$, compare $\{A1, A2, B1\}$ and $\{A3, B2, C1\}$.
- If $\{A1, A2, B1\} > \{A3, B2, C1\}$, then compare $\{A1\}$ and $\{A2\}$.
- If $\{A1, A2, B1\} < \{A3, B2, C1\}$, then compare $\{C1\}$ and $\{A3\}$.
- If $\{A1, A2, B1\} = \{A3, B2, C1\}$, then compare $\{B3\}$ and $\{B4\}$.

What if there are 13 coins?

More challenging problems

- How many weightings is needed to find a different coin from k given coins.

More challenging problems

- How many weightings is needed to find a different coin from k given coins.
- What if there are two lighter / different coins?

More challenging problems

- How many weightings is needed to find a different coin from k given coins.
- What if there are two lighter / different coins?
- What if there are three lighter / different coins?

More challenging problems

- How many weightings is needed to find a different coin from k given coins.
- What if there are two lighter / different coins?
- What if there are three lighter / different coins?

The end!