Interlacing Inequalities for Totally Nonnegative Matrices

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Dedicated to Professor T. Ando on the occasion of his 70th birthday.

Abstract

Suppose $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ are the eigenvalues of an $n \times n$ totally nonnegative matrix, and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$ are the eigenvalues of a $k \times k$ principal submatrix. A short proof is given of the interlacing inequalities:

$$\lambda_i \geq \tilde{\lambda}_i \geq \lambda_{i+n-k}, \; i = 1, \ldots, k.$$

It is shown that if $k = 1, 2, n-2, n-1$, λ_i and $\tilde{\lambda}_j$ are nonnegative numbers satisfying the above inequalities, then there exists a totally nonnegative matrix with eigenvalues λ_i and a submatrix with eigenvalues $\tilde{\lambda}_j$. For other values of k, such a result does not hold. Similar results for totally positive and oscillatory matrices are also considered.

1 Introduction

Let A be an $n \times n$ nonnegative matrix. It is totally nonnegative (TN) if all of its minors are nonnegative; it is totally positive (TP) if all of its minors are positive; it is oscillatory (OS) if A is TN and A^m is TP for some positive integer m. Evidently,

$$\text{TP} \;\subseteq \text{OS} \;\subseteq \text{TN} \;.$$

It is known (see [1, 8, 12]) that the inclusions are all strict, and that the closure of TP is TN.

Many authors, motivated by theory and applications, have studied TN, OS, and TP matrices (see for example [1, 3, 4, 7, 8, 9, 10, 11, 12, 15, 17]). These classes of matrices have a lot of nice properties that resemble those of positive semi-definite Hermitian matrices. For instance, positive semi-definite Hermitian matrices have nonnegative eigenvalues and so do TN, OS, and TP matrices. In fact, if a matrix is TP or OS, then it has positive distinct eigenvalues.

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Another interesting result on positive semi-definite (or general complex Hermitian or real symmetric) matrices is the interlacing theorem (see [2, 6]) asserting that:

(I) If A is an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, B is a $k \times k$ principal submatrix of A with eigenvalues $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$, then

$$\lambda_i \geq \tilde{\lambda}_i \geq \lambda_{i+n-k}, \ i = 1, \ldots, k. \tag{1.1}$$

(II) If $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$ satisfy (1.1), then there exists a Hermitian matrix A with eigenvalues λ_i so that A has a submatrix B with eigenvalues $\tilde{\lambda}_j$.

For TN, OS and TP matrices it is known [7, 16] that for principal submatrices in non-consecutive rows and columns the interlacing results are weaker. For example, if $k = n - 1$ and the submatrix is neither obtained from A by removing the first row and column, nor obtained from A by removing the last row and column, one only gets

$$\lambda_{j-1} \geq \tilde{\lambda}_j \geq \lambda_{j+1}, \qquad 1 \leq j \leq n-1,$$

where $\lambda_0 = \lambda_1$.

The situation for principal submatrices lying in *consecutive* rows and columns is nicer. In this case it is known [8, Theorem 14] (see also [1]) that (I) holds for TN, OS, and TP matrices. The purpose of this paper is to study to what extent is (II) true for TN, OS, TP matrices. In particular, we show that (II) does not hold for TP, OS, TP matrices if $k \neq 1, 2, n-2, n-1$. Furthermore, (II) indeed holds for TN matrices if $k = 1, 2, n-2, n-1$, for OS matrices if $k = 1, n-1$, and for TP matrices if $k = 1$. The cases for OS matrices when $k = 2, n-2$, and for TP matrices when $k = 2, n-2, n-1$, are still open.

In the next section, we will give a short proof for the interlacing inequalities (1.1) for TN, OS, TP matrices. We also obtain interlacing inequalities for the Schur complements of these matrices. Section 3 will be devoted to the study of the converse theorem.

The following results will be used in our discussion. Their proofs can be found in [1].

Lemma 1.1 *Let $A = (a_{ij})$ be an $n \times n$* TN *matrix.*
(a) *Suppose $P = (p_{ij})$ is such that $p_{ij} = 1$ if $i + j = n + 1$ and 0 otherwise. Then PAP is* TN *.*
(b) *If A is invertible and $D = \mathrm{diag}(1, -1, 1, -1, \ldots)$, then $DA^{-1}D$ is* TN *.*
(c) *Suppose that A has non zero row or column and $a_{ij} = 0$. If $j \geq i$ then $a_{kl} = 0$ for all (k, l) with $k \leq i$ and $l \geq j$. If $j \leq i$ then $a_{kl} = 0$ for all (k, l) with $k \geq i$ and $l \leq j$.*

Lemma 1.2 *Let A be an invertible $n \times n$* TN *matrix. The following conditions are equivalent.*
(a) *A is* OS *.*
(b) *$a_{ij} > 0$ whenever $|i - j| \leq 1$.*
(c) *A is irreducible.*
(d) *Neither A nor A^t is of the form $\begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}$ with A_1 and A_3 square.*
(e) *A^{n-1} is TP.*

2 Interlacing Inequalities

We give a short proof for the interlacing inequality. Our proof is more elementary compared with those in [8, Theorem 14] and [1, Theorem 6.5].

Theorem 2.1 *Let A be* TN *with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Suppose $\tilde{A}$ is a $k \times k$ principal submatrix of A lying in rows and columns with consecutive indices and has eigenvalues $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$. Then*

$$\lambda_i \geq \tilde{\lambda}_i \geq \lambda_{i+n-k}, \ i = 1, \ldots, k.$$

Furthermore, if A is TP *or* OS *, then all the inequalities are strict.*

Proof. It suffices to prove the case when A is TP . One can then apply continuity arguments to get the conclusion for TN matrices. Also, we can focus on the special case when $k = n-1$, and then apply the special result $(n-k)$ times to get the general result.

By Lemma 1.1, we may assume that $\tilde{A}$ is obtained from A by removing the first row and first column. By the result in [17], we can apply a sequence of elementary row operations to A by subtracting a suitable multiple of the jth row from the $(j+1)$st row, for $j = 2, \ldots, n$, to eliminate the nonzero entries in the $(3,1), \ldots, (n,1)$ positions, so that the resulting matrix is still TN. Then, we can eliminate the nonzero entries in the $(4,2), \ldots, (n,2)$ positions, and so forth, until we get a matrix with zero entries in the (i,j) positions whenever $i \geq j+2$. Multiplying all the elementary matrices used to do the elimination, we get a matrix S of the form $[1] \oplus S_1$, where S_1 is lower triangular. (Note that the first row of A is unchanged during the elimination process. This is why S is of the asserted form.) Since the inverse of each elementary matrix used for the reduction is TN, we see that $B = SAS^{-1}$ is TN. Now, applying a similar argument to B^t, we can eliminate the nonzero entries of B^t in the (i,j) positions whenever $i \geq j+2$. Let $\tilde{S}$ be the product of all elementary matrices used to do the elimination. Then $\tilde{S}$ is of the form $[1] \oplus \tilde{S}$, where $\tilde{S}$ is lower triangular. It is easy to check that $\tilde{S}B^t\tilde{S}^{-1}$ has the desired zero entries in the lower triangular part, and the zero entries in the upper triangular part of B^t are not disturbed. Hence, $T = \tilde{S}(SAS^{-1})^t\tilde{S}^{-1}$ is in tridiagonal form. We can further apply a diagonal similarity to make T a symmetric matrix. If we remove the first row and first column of T, we get $T_1 = \tilde{S}_1(S_1\tilde{A}S_1^{-1})^t\tilde{S}_1^{-1}$. Now, by the interlacing inequalities on real symmetric matrices, the eigenvalues of T_1 interlace those of $\tilde{S}(SAS^{-1})^t\tilde{S}^{-1}$, which are the same as those of A. The result follows. □

Next we consider interlacing inequalities for the Schur complement of a TN matrix.

Theorem 2.2 *Let A be an $n \times n$* TN *matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Suppose $\alpha = \{1, \ldots, k\}$ or $\{n-k+1, n-k+2, \ldots, n\}$ such that $A[\alpha']$ is invertible and A/α' has eigenvalues $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$. Then*

$$\lambda_i \geq \tilde{\lambda}_i \geq \lambda_{i+n-k}, \quad 1 \leq i \leq k.$$

If A is OS *or* TP *, then all the inequalities are strict.*

Proof. First consider the case when A is OS, which may be TP. Note that A/α' is a $k \times k$ principal submatrix of A^{-1}. Since $B = DA^{-1}D$ is OS if $D = \text{diag}(1, -1, 1, -1, \ldots)$, the interlacing theorem holds for B and hence for A^{-1}. It follows that

$$1/\lambda_i < 1/\tilde{\lambda}_i < 1/\lambda_{i+n-k}, \quad 1 \le i \le k.$$

Taking reciprocals we get the desired result. A continuity argument extends the result to TN matrices. □

3 Converse Theorems

In this section, we study the converse of the interlacing theorem. Since the interlacing inequalities for TN , OS , and TP matrices only hold for submatrices lying in consecutive rows and columns, we shall focus on these cases. We restate the question as follows:

Suppose we have nonnegative numbers $\lambda_1 \ge \cdots \ge \lambda_n$ and $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_k$ satisfying (1.1). Is there a TN (OS *or* TP) *matrix A with a $k \times k$ principal submatrix $\tilde{A}$ lying in rows and columns with consecutive indices so that A and $\tilde{A}$ have eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$ and $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_k$, respectively?* (*For OS and TP matrices, it is understood that all the inequalities are strict.*)

We first consider TN matrices.

Theorem 3.1 *Suppose $k = 1, 2, n-2$ or $n-1$, and suppose $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$ and $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_k \ge 0$ satisfy* (1.1). *Then there is a* TN *matrix A with a $k \times k$ principal submatrix $\tilde{A}$ lying in rows and columns with consecutive indices so that A and $\tilde{A}$ have eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$ and $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_k$, respectively.*

Proof. Suppose λ_i and $\tilde{\lambda}_j$ are given that satisfy the interlacing inequalities (1.1). We may assume that all the inequalities are strict and use continuity arguments to get the conclusion for the general case.

For $k = 1, n-1$, let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$ such that the leading $k \times k$ principal submatrix has eigenvalues $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_k$. Then one can apply a finite number of Householder transforms to get an invertible S such that

(a) $S = [1] \oplus S_1$ if $k = 1$,

(b) $S = S_1 \oplus [1]$ if $k = n-1$,

and $\hat{A} = S^{-1}AS$ is tridiagonal. Applying a diagonal similarity transform to $\hat{A}$ if necessary, we may assume that the tridiagonal matrix $\hat{A}$ is nonnegative. The resulting matrix is the desired TN matrix.

Suppose $k = 2$. If $n = 3$, we are done by the previous construction as $k = n-1$. If $n \ge 4$, then we can use the procedures of the preceding paragraph to construct a 2×2 TN matrix A_1 with eigenvalues λ_1, λ_{n-1} and $(2,2)$ entry equal to $\tilde{\lambda}_1$, and then construct a $(n-2) \times (n-2)$ TN matrix A_2 with eigenvalues $\lambda_2 \ge \cdots \ge \lambda_{n-2} \ge \lambda_n$ and $(1,1)$ entry equal to $\tilde{\lambda}_2$. Then the submatrix of $A = A_1 \oplus A_2$ lying in rows and columns 2 and 3, is diagonal and has eigenvalues $\tilde{\lambda}_1 \geq \tilde{\lambda}_2$.

Suppose $k = n-2$. Divide the numbers $\lambda_1 \geq \cdots \geq \lambda_n$ and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_{n-2}$ into two collections $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows.

Step 1. Put λ_1 in $\mathcal{C}_1$ and put λ_n in $\mathcal{C}_2$.

Step 2. For $2 \leq i \leq n-2$, by the interlacing inequalities $\tilde{\lambda}_i \in [\lambda_{i+2}, \lambda_i]$. If $\tilde{\lambda}_i \geq \lambda_{i+1}$, put both $\tilde{\lambda}_i$ and λ_{i+1} in $\mathcal{C}_1$. Otherwise, put them in $\mathcal{C}_2$.

By this construction, the numbers in $\mathcal{C}_1$ will be of the form:

$$\lambda_1 \geq \tilde{\lambda}_{i_1} \geq \lambda_{i_1+1} \geq \tilde{\lambda}_{i_2} \geq \lambda_{i_2+1} \geq \cdots,$$

and the numbers in $\mathcal{C}_2$ will be of the form

$$\lambda_n \leq \tilde{\lambda}_{j_1} < \lambda_{j_1+1} \leq \tilde{\lambda}_{j_2} < \lambda_{j_2+1} \leq \cdots.$$

One can now construct A_1 with eigenvalues $\lambda_1, \lambda_{i_1+1}, \lambda_{i_2+1}, \ldots$ in $\mathcal{C}_1$ so that the submatrix of A_1 obtained by removing the first row and first column has eigenvalues $\tilde{\lambda}_{i_1}, \tilde{\lambda}_{i_2}, \ldots$, in $\mathcal{C}_1$. Similarly, one can construct A_2 with eigenvalues $\lambda_n, \lambda_{j_1+1}, \lambda_{j_2+1}, \ldots$ in $\mathcal{C}_2$ so that the submatrix of A_2 obtained by removing the last row and last column has eigenvalues $\tilde{\lambda}_{i_1}, \tilde{\lambda}_{i_2}, \ldots$, in $\mathcal{C}_2$. Let $A = A_1 \oplus A_2$. The principal submatrix of A in rows and columns $2, \ldots, n-1$ has eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-2}$. $\square$

In the cases $k = 2$ and $k = n-2$ the principal submatrix was neither leading nor trailing. One may wonder whether it is possible to construct a TN matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a leading or trailing submatrix with eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k$. We will see that it is not possible. We will also show that the converse of interlacing theorem for TN matrices fails in general if $2 < k < n-2$. We first establish the following lemma.

Lemma 3.2 *Let A be an invertible TN matrix. Suppose that $\tilde{\lambda}$ is a repeated eigenvalue of a leading principal submatrix of A. Then $\tilde{\lambda}$ is an eigenvalue of A.*

Proof. We prove the result by induction. The statement is clear if $n = 2$. Suppose that $n > 2$ and that the result is true for TN matrices of order less than n. Suppose A is an $n \times n$ TN matrix and B is a leading principal submatrix of A with a repeated eigenvalue $\tilde{\lambda}$. Since A is invertible, all eigenvalues of A are positive. By Theorem 2.1, B also has positive eigenvalues and hence B is invertible. Note that B must be reducible by Lemma 1.2; otherwise, B is oscillatory and has distinct positive eigenvalues. Suppose $B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$. Then $A = \begin{pmatrix} B_1 & 0 \\ A_2 & A_3 \end{pmatrix}$ by Lemma 1.1 (c), and B_3 is the left top corner of A_3. If $\tilde{\lambda}$ is an eigenvalue of B_1, then it is an eigenvalue of A. If $\tilde{\lambda}$ is not an eigenvalue of B_1 then it is a repeated eigenvalue of B_3, which is a leading principal submatrix of A_3. By the induction assumption A_3 has $\tilde{\lambda}$ as an eigenvalue. $\square$

Now consider the converse theorem for TN matrices with $n = 4$ and $k = 2 = n-2$. The data $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 2$, $\lambda_4 = 1$ and $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 2.5$ satisfies the interlacing conditions. Since we require the submatrix to have a repeated eigenvalue which is not an eigenvalue of the full matrix the submatrix cannot be leading or trailing.

Theorem 3.3 *Suppose $n-2 > k > 2$. Let $\lambda_i = n-i+1$ for $i = 1, \ldots, n$. Let $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = n - 5/2$, and $\tilde{\lambda}_j = n-j+1/2$ for $j = 4, \ldots, k$. Then the λ_i's and $\tilde{\lambda}_j$'s satisfy the interlacing inequalities (1.1), but there is no* TN *matrix A with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and a principal submatrix B lying in consecutive rows and columns having eigenvalues $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$.*

Proof. Suppose k, n, $\lambda_1 \geq \cdots \geq \lambda_n$ and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$ satisfy the hypotheses. One readily checks that the interlacing inequalities (1.1) are satisfied.

Suppose there is a TN matrix A with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and a principal submatrix B lying in consecutive rows and columns having eigenvalues $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$. Then B must be reducible. Otherwise, B is oscillatory and has distinct eigenvalues. Suppose $B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$. Then $A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}$ so that B_1 is the right bottom corner of A_1, and B_3 is the left top corner of A_3. Moreover, either B_1 or B_3 has $\tilde{\lambda}_1$ as a repeated eigenvalue. If B_3 does, then A_3 is a TN matrix with positive eigenvalues whose leading principal submatrix has a repeated eigenvalue. This contradicts the previous lemma. If B_1 does, then PA_1P is a matrix contradicting the previous lemma, where P is the anti-diagonal matrix with all anti-diagonal entries equal to one. □

We have shown the converse interlacing theorem holds for TN matrices if and only if $k = 1, 2, n-2, n-1$. Next we show that the converse interlacing theorem fails for TP and OS matrices in general if $2 < k < n-2$. This is not an immediate consequence of the result for TN matrices because in the TP and OS cases we assume that all the inequalities are strict. We need the following lemma.

Lemma 3.4 *Suppose A is positive and all 2×2 minors of A are positive. In addition, if $a_{ii} \in [\alpha, \beta]$ for all $i = 1, \ldots, n$, and $a_{i,i+1} = a_{i+1,i}$ for $i = 1, \ldots, n-1$. Then $a_{ij} < \alpha(\beta/\alpha)^{|i-j|}$ for all $i \neq j$.*

Proof. We prove the statement by induction on $|i-j|$. Suppose $i - j = 1$. Since $a_{ii}a_{jj} > a_{ij}a_{ji} = a_{ij}^2$, the result follows.

Suppose the result is true for $|i-j| < k$ with $1 < k < n-1$. Consider a_{ij} with $j - i = k$ and the submatrix $\tilde{A} = \begin{pmatrix} a_{i,i+1} & a_{ij} \\ a_{i+1,i+1} & a_{i+1,j} \end{pmatrix}$. We have

$$\alpha \leq a_{i+1,i+1} \leq \beta, \ a_{i,i+1} \leq \alpha(\beta/\alpha), \text{ and } a_{i+1,j} \leq \alpha(\beta/\alpha)^{k-1}$$

by the induction assumption. Since $\alpha \leq a_{i+1,i+1} \leq \beta$ and $\det(\tilde{A}) > 0$, we have

$$\alpha(\beta/\alpha)\alpha(\beta/\alpha)^{k-1} \geq a_{i,i+1}a_{i+1,j} > a_{i+1,i+1}a_{ij} \geq \alpha a_{ij}.$$

The result follows. $\square$

Theorem 3.5 *Suppose $n-2>k>2$. Then there are distinct λ_i's and $\tilde{\lambda}_j$'s satisfy the interlacing inequalities (1.1) strictly, such that there is no* OS *or* TP *matrix A with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and a principal submatrix B lying in consecutive rows and columns having eigenvalues $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$.*

Proof. If the converse of interlacing theorem holds for TP or OS matrices for k with $2<k<n-2$, then we can extend the result to TN matrices by the following argument.

Given data $\lambda_1 \geq \cdots \geq \lambda_n$ and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_k$ satisfying the interlacing inequalities, we can perturb the data so that strict inequalities hold, and construct OS matrices attaining the perturbed data $\lambda_i(t)$ and $\tilde{\lambda}_j(t)$. Further, we can apply a diagonal similarity to the constructed matrix $A(t)$ so that the $(i,i+1)$ and $(i+1,i)$ entries of these matrices are equal for all $i=1,\dots,n-1$. By Lemma 3.4, these matrices are bounded. We can then choose a sequence of $A(t)$ whose eigenvalues approach to λ_i and a principal submatrix (with fixed consecutive row and column indices) having eigenvalues approaching $\tilde{\lambda}_j$. Since this is a bounded sequence of matrices, we can find a limit point A, which will be a TN matrix with the eigenvalues $\lambda_1,\dots,\lambda_n$ and submatrix with eigenvalues $\tilde{\lambda}_1,\dots,\tilde{\lambda}_k$. However, since the result is not valid for TN matrices, we see that the converse of the interlacing theorem for OS or TP matrices cannot always hold. $\square$

Now we turn our attention to positive results in the OS and TP cases. In such cases, we always assume that

$$\lambda_1 > \cdots > \lambda_n > 0, \quad \tilde{\lambda}_1 > \cdots > \tilde{\lambda}_k,$$

$$\lambda_i > \tilde{\lambda}_i > \lambda_{i+n-k}, \quad i=1,\dots,k.$$

Let $\hat{A}$ be the TN tridiagonal matrix constructed in the proof of Theorem 3.1 with eigenvalues $\lambda_1,\dots,\lambda_n$ with leading principal submatrix having eigenvalues $\tilde{\lambda}_1,\dots,\tilde{\lambda}_{n-1}$. Since $\hat{A}$ and its leading principal submatrix have no common eigenvalues the sub- and super-diagonal entries of $\hat{A}$ must be non-zero. Lemma 1.2 tells us that $\hat{A}$ is in fact OS.

Next, we establish the converse theorem for OS and TP matrices when $k=1$. Clearly, the latter case actually covers the former case. Nevertheless, we will present very different proofs for the two cases, in the hope that the techniques and ideas presented may be useful in solving the unknown cases.

Here we show that one can construct an oscillatory tridiagonal with eigenvalues $\lambda_1,\dots,\lambda_n$ and $(1,1)$ entry $\tilde{\lambda}_1$ if $\lambda_1 > \tilde{\lambda}_1 > \lambda_n$. To this end, we let m be such that $\lambda_m \geq \tilde{\lambda}_1 > \lambda_{m+1}$, and let $\eta = \lambda_m + \lambda_{m+1} - \tilde{\lambda}_1$ If $m=1$, then $\eta > \lambda_{m+1}$, and one can find a sufficiently small $\varepsilon > 0$ so that the numbers $\mu_1 = \eta - (n-2)\varepsilon$ and $\mu_i = \lambda_{i+1} + \varepsilon$ for $i=2,\dots,n-1$, satisfy

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{n-1} > \lambda_n. \tag{3.1}$$

If $m>1$, then $\eta \geq \lambda_{m+1}$, and one can find sufficiently small $\varepsilon_1, \varepsilon_2 > 0$ so that $(m-1)\varepsilon_1 = (n-m)\varepsilon_2$, and the numbers $\mu_i = \lambda_i - \varepsilon_1$ for $i=1,\dots,m-1$, $\mu_m = \eta + \varepsilon_2$, $\mu_i = \lambda_{i+1} + \varepsilon_2$

for $i = m+1, \ldots, n-1$, satisfy (3.1). In both cases, we can use the result in the preceding paragraph to conclude that there is a tridiagonal OS matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and the submatrix obtained by removing the first row and column has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. By construction, the $(1,1)$ entry of this matrix is

$$\sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i = \tilde{\lambda}_1$$

as desired.

Finally, we show that the converse theorem holds for $k = 1$ for TP matrices. Given $\lambda_1 > \cdots > \lambda_n$, let $T(0)$ be symmetric, oscillatory and tridiagonal with eigenvalues $\lambda_1^{1/n}, \ldots, \lambda_n^{1/n}$. Let $T(t)$ be the solution to the QR flow [5, Section 5.5]. Then, $T(t)$ is a continuous isospectral family of tridiagonal matrices. One can show that the sub- and super-diagonal entries of $T(t)$ remain non-zero for all $t \in R$, and so, by continuity these entries must be positive. Thus $T(t)$ is oscillatory for all $t \in R$. It is known that $T(1), T(2), \ldots$ are the iterates generated by the QR iteration applied to $T(0)$, thus,

$$\lim_{j \to \infty,\ j \in N} T(j) = \operatorname{diag}(\lambda_1^{1/n}, \ldots, \lambda_n^{1/n}).$$

For a proof of this fact see for example [14, Theorem 8.6.1], but note that Parlett considers the QL iteration, and orders the eigenvalues in increasing order.

Also, $T(-1), T(-2), \ldots$ are the iterates generated by the QR iteration run in reverse applied to $T(0)$, thus

$$\lim_{j \to -\infty,\ j \in N} T(j) = \operatorname{diag}(\lambda_n^{1/n}, \ldots, \lambda_1^{1/n}).$$

See [5, Corollary 5.4] for a discussion of the QR iteration run in reverse.

Let $A(t) = T(t)^n$. Then, since $T(t)$ is $n \times n$ and OS, $A(t)$ is a continuous isospectral family of TP matrices. By the continuity of matrix multiplication

$$\lim_{j \to -\infty,\ j \in N} A(j) = \operatorname{diag}(\lambda_n, \ldots, \lambda_1), \quad \text{and} \quad \lim_{j \to \infty,\ j \in N} A(j) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

So, $A(t)_{11}$ takes on every value $\tilde{\lambda}_1$ in (λ_n, λ_1) as t varies over R. This establishes the converse theorem for TP for $k = 1$.

Remark 3.6 One can easily obtain the results corresponding to the converse of the interlacing theorem on the Schur complements of TN, OS, and TP matrices.

Remark 3.7 It is open whether the converse theorem holds for TP matrices if $k = 2, n-2, n-1$ and for OS matrices if $k = 2, n-2$.

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