Numerical radius isometries

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Abstract

Let V be a direct sum of full matrix algebras, or the algebra of block upper triangular matrices. Suppose $r(A)$ is the numerical radius of $A \in V$. We characterize mappings $f : V \to V$ that satisfy $r(f(A) - f(B)) = r(A - B)$ for all $A, B \in V$.

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1   Introduction

Let M_n be the algebra of $n \times n$ complex matrices. Suppose $n_1, \ldots, n_k$ are positive integers such that $n = n_1 + \cdots + n_k$. Let $D(n_1, \ldots, n_k)$ be the subalgebra of M_n consisting of matrices of the form $A_1 \oplus \cdots \oplus A_k$ with $A_j \in M_{n_j}$ for $j = 1, \ldots, k$, and let $T(n_1, \ldots, n_k)$ be the subalgebra of M_n consisting of block upper triangular matrices $A = (A_{ij})_{1 \le i,j \le k}$ such that $A_{jj} \in M_{n_j}$. Note that up to unitary similarity (an isometric isomorphism), these are all the finite dimensional C^*-algebras and nest algebras. The *numerical range* and *numerical radius* of $A \in M_n$ are defined by

$$W(A) = \{v^* A v : v \in \mathbf{C}^n, \|v\| = 1\} \quad \text{and} \quad r(A) = \max\{|z| : z \in W(A)\},$$

respectively. A mapping $f : V \to V$ is a *numerical radius isometry* if

$$r(f(A) - f(B)) = r(A - B) \quad \text{for all } A, B \in V.$$

The above definition does not require f to be linear. Nevertheless, the result of Charzyński [1] ensures that the mapping $L : V \to V$ defined by $L(A) = f(A) - f(0)$ is real linear such that $r(L(A)) = r(A)$ for all $A \in V$. Thus, the problem of characterizing numerical radius isometries reduces to studying real linear numerical radius isometries on V. The purpose of this note is to characterize real linear numerical radius isometries on V.

2 Results and Proofs

Denote by $\{e_1, \ldots, e_n\}$ the standard basis for $\mathbf{C}^n$, and $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for M_n. We always assume that $n_1, \ldots, n_k$ are positive integers such that $n = n_1 + \cdots + n_k$, and $V = D(n_1, \ldots, n_k)$ or $T(n_1, \ldots, n_k)$.

First, we prove that if L is a numerical radius isometry on V satisfying $L(0) = 0$ then there exist complex units $\xi, \xi_1, \ldots, \xi_k$ such that

$$L(I) = \begin{cases} \xi I & \text{if } V = T(n_1, \ldots, n_k), \\ \xi_1 I_{n_1} \oplus \cdots \oplus \xi_k I_{n_k} & \text{if } V = D(n_1, \ldots, n_k). \end{cases}$$

To achieve this, we need the following lemma.

Lemma 1 *Let $V = M_n$ or $T(n_1, \ldots, n_k)$. Suppose $H, K \in V$ are (real) linearly independent such that $r(xH + yK) \leq 1$ whenever $x, y \in \mathbf{R}$ satisfy $x^2 + y^2 \leq 1$. The following conditions are equivalent.*

(a) *There exists a complex unit μ such that $(H, K) = \mu(I, \pm iI)$.*

(b) *For any $A \in V$ there exist $x, y \in \mathbf{R}$ such that $x^2 + y^2 = 1$ and $r(xH + yK + A) = 1 + r(A)$.*

Proof. The implication (a) $\Rightarrow$ (b) is clear. We consider the converse. Assume (b) holds. Let $A_\theta = (\cos\theta + i\sin\theta)E_{11}$. Then there exist $x_\theta, y_\theta \in \mathbf{R}$ such that $x_\theta^2 + y_\theta^2 = 1$ and

$$r(x_\theta H + y_\theta K + A_\theta) = 1 + r(A_\theta) = 2.$$

Thus, there exists a unit vector v_θ such that

$$2 = |v_\theta^*(x_\theta H + y_\theta K + A_\theta)v_\theta| = |v_\theta^*(x_\theta H + y_\theta K)v_\theta| + |v_\theta^* A_\theta v_\theta|.$$

It follows that $v_\theta = \eta_\theta e_1$ for some complex unit η_θ, and

$$e_1^*(x_\theta H + y_\theta K)e_1 = \cos\theta + i\sin\theta. \tag{1}$$

Suppose H and K have $(1,1)$ entries equal to $h_1 + ih_2$ and $k_1 + ik_2$, respectively, with $h_1, h_2, k_1, k_2 \in \mathbf{R}$. Let

$$C = \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix}.$$

By (1), for any $\theta \in [0, 2\pi)$, there exists a unit vector $u_\theta \in \mathbf{R}^2$ such that $Cu_\theta = (\cos\theta, \sin\theta)^t$. Hence C maps the unit ball in $\mathbf{R}^2$ onto itself, and thus C is an isometry on $\mathbf{R}^2$ and is of the form

$$C = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad or \quad \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}.$$

Hence, $z_1 = h_1 + ih_2$ is a complex unit, and $k_1 + ik_2 = i\delta_1 z_1$ where $\delta_1 \in \{1, -1\}$. Applying the same argument to the (j, j) entries of H and K, we see that they must be of the form z_j and $i\delta_j z_j$ with $|z_j| = 1$ and $\delta_j \in \{1, -1\}$.

Next, we show that all the diagonal entries of H (respectively, K) are the same. Without loss of generality, we assume that $z_1 = 1$ and $\delta_1 = 1$; otherwise, replace (H, K) by $(H/z_1, \pm K/z_1)$.

Suppose

$$H = \begin{pmatrix} 1 & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \quad and \quad K = i \begin{pmatrix} 1 & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Observe that for any $B \in M_n$ with $r(B) = 1$, the largest eigenvalue of $(B + B^*)/2$ is less than or equal to one. As a result, if $B \in M_n$ has numerical radius and $(1,1)$ entry both equal to one, then $(B + B^*)/2 = [1] \oplus B_1$, and hence B is of the form

$$B = \begin{pmatrix} 1 & u \\ -u^* & * \end{pmatrix}.$$

Now, applying this observation to the matrix

$$Z_\theta = (\cos\theta H + \sin\theta K)/(\cos\theta + i\sin\theta), \qquad \theta \in \mathbf{R},$$

whose $(1,1)$ entry and numerical radius both equal 1, we see that

$$\frac{\cos\theta H_{12} + i\sin\theta K_{12}}{\cos\theta + i\sin\theta} = -\left\{ \frac{\cos\theta H_{21} + i\sin\theta K_{21}}{\cos\theta + i\sin\theta} \right\}^*,$$

i.e.,

$$\cos^2\theta H_{12} + \sin^2\theta K_{12} + i\cos\theta\sin\theta(K_{12} - H_{12})$$
$$= -\cos^2\theta H_{21}^* - \sin^2\theta K_{21}^* + i\cos\theta\sin\theta(K_{21}^* - H_{21}^*).$$

It follows that

$$H_{12} = -H_{21}^*, \quad K_{12} = -K_{21}^*, \quad K_{12} - H_{12} = K_{21}^* - H_{21}^*,$$

and hence, $K_{12} = H_{12} = -H_{21}^* = -K_{21}^*$.

Now, for each $j > 1$ and $\theta \in \mathbf{R}$, let

$$Z_{\theta,j} = \cos\theta \begin{pmatrix} 1 & \mu_j \\ -\bar\mu_j & z_j \end{pmatrix} + i\sin\theta \begin{pmatrix} 1 & \mu_j \\ -\bar\mu_j & \delta_j z_j \end{pmatrix} = e^{i\theta} \begin{pmatrix} 1 & \mu_j \\ -\bar\mu_j & z_j e^{-i\theta}(\cos\theta + i\delta_j \sin\theta) \end{pmatrix}$$

be the 2×2 submatrix of $\cos\theta H + \sin\theta K$ lying in the first and jth rows and columns.

Suppose $\mu_j \neq 0$. If $\delta_j = -1$, then there exists $\theta \in \mathbf{R}$ such that $\overline{(\bar z_j \mu_j e^{i2\theta})} \neq \bar z_j \bar\mu_j e^{i2\theta}$; thus the matrix

$$\bar z_j e^{i\theta} Z_{\theta,j} = \begin{pmatrix} \bar z_j e^{i2\theta} & \bar z_j \mu_j e^{i2\theta} \\ -\bar z_j \bar\mu_j e^{i2\theta} & 1 \end{pmatrix}$$

has numerical radius one and its Hermitian part has an eigenvalue larger than one, which is a contradiction. Hence, we see that $\delta_j = 1$ and

$$Z_{\theta,j} = e^{i\theta} \begin{pmatrix} 1 & \mu_j \\ -\bar\mu_j & z_j \end{pmatrix}.$$

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Now, let $\xi \in \mathbf{C}$ such that $\mu_j \xi = |\mu_j|$ and $B_\xi = \xi(E_{11} + E_{jj}) + 2E_{1j} \in V$. Then $r(B_\xi) = 2$, and there exist a unit vector $(\cos\psi, \sin\psi)^t \in \mathbf{R}^2$ and a unit vector $v \in \mathbf{C}^n$ such that

$$
\begin{aligned}
3 &= r(\cos\psi H + \sin\psi K + B_\xi) \\
&= |v^*(\cos\psi H + \sin\psi K + B_\xi)v| \\
&= |v^*(\cos\psi H + \sin\psi K)v| + |v^* B_\xi v|.
\end{aligned}
$$

Note that $|v^* B_\xi v| = 2$ if and only if $v = \mu(e_1 + \xi e_j)$ for some $\mu \in \mathbf{C}$ with $|\mu| = 1/\sqrt{2}$. Hence,

$$
1 = |v^*(\cos\psi H + \sin\psi K)v| = \frac{1}{2}\left|(1,\bar{\xi})\begin{pmatrix} 1 & \mu_j \\ -\bar{\mu}_j & z_j \end{pmatrix}\begin{pmatrix} 1 \\ \xi \end{pmatrix}\right| = |(1+z_j)/2|,
$$

i.e., $z_j = 1$.

Suppose $\mu_j = 0$. For $s \in \{1, i\}$, let

$$
B_s = s(E_{11} + E_{jj}) + 2E_{1j}.
$$

Then $r(B_s) = 2$. So, there exist a unit vector $(x_s, y_s) \in \mathbf{R}^2$ and a unit vector $v \in \mathbf{C}^n$ such that

$$
3 = r(x_s H + y_s K + B_s) = |v^*(x_s H + y_s K + B_s)v| = |v^*(x_s H + y_s K)v| + |v^* B_s v|.
$$

Note that $|v^* B_s v| = 2$ if and only if $v = v_s = \mu(e_1 + s e_j)$ for some $\mu \in \mathbf{C}$ with $|\mu| = 1/\sqrt{2}$. Putting $s = 1$, we have

$$
1 = v_s^*(x_s H + y_s K)v_s = \{(x_s + iy_s) + z_j(x_s + i\delta_j y_s)\}/2.
$$

Hence the two complex units $(x_s + iy_s)$, $z_j(x_s + i\delta_j y_s)$ must both equal one, i.e., $(x_s, y_s) = (1, 0)$ and $z_j = 1$. Putting $s = i$, we have

$$
i = v_s^*(x_s H + y_s K)v_s = \{(x_s + iy_s) + (x_s + i\delta_j y_s)\}/2.
$$

Thus, $(x_s, y_s) = (0, 1)$ and the (j, j) entry of K must be i.

Now, we see that all diagonal entries of H equal one. Since $r(H) = 1$, it follows that $(H + H^*)/2 = I$. Furthermore, the skew-Hermitian part $(H - H^*)/(2i)$ must be zero, otherwise, there exists a unit vector $v \in \mathbf{C}^n$ such that $v^*(H - H^*)v = id$ for some nonzero $d \in \mathbf{R}$ and thus $|v^* Hv| = \sqrt{1 + d^2/4} > 1 = r(H)$. Thus, $H = I$. We have also proved that all diagonal entries of K equal i, and consequently, $K = iI$. ∎

Corollary 2 *Let $V = D(n_1, \ldots, n_k)$. Suppose $H, K \in V$ are (real) linearly independent such that $r(xH + yK) \le 1$ whenever $x, y \in \mathbf{R}$ satisfy $x^2 + y^2 \le 1$. The following conditions are equivalent.*

(a) *For any $A \in V$ there exist $x, y \in \mathbf{R}$ such that $x^2 + y^2 = 1$ and $r(xH + yK + A) = 1 + r(A)$.*

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(b) *There exist complex units $\mu_1, \ldots, \mu_k$, and $d_1, \ldots, d_k \in \{i, -i\}$ such that*

$$(H, K) = (\mu_1 I_{n_1} \oplus \cdots \oplus \mu_k I_{n_k}, d_1 \mu_1 I_{n_1} \oplus \cdots \oplus d_k \mu_k I_{n_k}).$$

Proof. Suppose $H = H_1 \oplus \cdots \oplus H_k$ and $K = K_1 \oplus \cdots \oplus K_k$ such that $H_j, K_j \in M_{n_j}$ for $j = 1, \ldots, k$. Suppose (a) holds. We can specialize condition (a) to those matrices $A = A_1 \oplus \cdots \oplus A_k \in D(n_1, \ldots, n_k)$ such that $A_i = 0_{n_i}$ except for $i = j$. Then we can apply Lemma 1 to conclude that there exist a complex unit μ_j and $d_j \in \{i, -i\}$ such that $(H_j, K_j) = (\mu_j I_{n_j}, d_j \mu_j I_{n_j})$. Thus, condition (b) holds.

Conversely, suppose (b) holds. Then for any $A = A_1 \oplus \cdots \oplus A_k \in D(n_1, \ldots, n_k)$, there exists $j \in \{1, \ldots, k\}$ such that $r(A) = r(A_j)$. By Lemma 1, there exist $x, y \in \mathbf{R}$ with $x^2 + y^2 = 1$ such that

$$1 + r(A) = 1 + r(A_j) = r(xH_j + yK_j + A_j) \le r(xH + yK + A) \le 1 + r(A).$$

Thus, condition (a) holds. ∎

Corollary 3 *Let $V = D(n_1, \ldots, n_k)$ or $T(n_1, \ldots, n_k)$. Suppose L is a real linear numerical radius isometry on V.*

(a) *If $V = T(n_1, \ldots, n_k)$, then there exists a complex unit μ such that $(L(I), L(iI)) = \mu(I, \pm iI)$.*

(b) *If $V = D(n_1, \ldots, n_k)$, then there exist complex units $\mu_1, \ldots, \mu_k$, and $d_1, \ldots, d_k \in \{i, -i\}$ such that $(L(I), L(iI)) = (\mu_1 I_{n_1} \oplus \cdots \oplus \mu_k I_{n_k}, d_1 \mu_1 I_{n_1} \oplus \cdots \oplus d_k \mu_k I_{n_k})$.*

Proof. Suppose $V = T(n_1, \ldots, n_k)$. Note that $(H, K) = (I, iI)$ satisfies Lemma 1 (b). By the assumption on L, the pair of matrices $(L(I), L(iI))$ also satisfies the same condition, and hence has the form given in Lemma 1 (a). One can use Corollary 2 and a similar argument to get the result if $V = D(n_1, \ldots, n_k)$. ∎

We need one more lemma to prove our main result.

Lemma 4 *Let $V = D(n_1, \ldots, n_k)$ or $T(n_1, \ldots, n_k)$. Suppose $F : V \to \mathcal{K}(\mathbf{C})$, where $\mathcal{K}(\mathbf{C})$ is the set of nonempty compact convex subsets of $\mathbf{C}$ such that*

$$F(\alpha A + \beta I) = \alpha F(A) + \beta, \quad \text{for any } \alpha, \beta \in \mathbf{C},$$

and suppose $f : V \to \mathbf{R}$ is defined by $f(A) = \max\{|z| : z \in F(A)\}$. If L is a real linear map L on V satisfying $(L(I), L(iI)) = (I, iI)$ and $f(L(A)) = f(A)$ for all $A \in V$, then $F(L(A)) = F(A)$ for all $A \in V$.

Proof. Let L satisfy the hypotheses. Suppose $A \in V$ is such that $F(L(A)) \ne F(A)$. If there is $\mu \in F(L(A)) \setminus F(A)$, then there exists $\eta \in \mathbf{C}$ such that

$$f(L(A - \eta I)) \ge |\mu - \eta| > \max_{z \in F(A)} |z - \eta| = f(A - \eta I),$$

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which is a contradiction. By reversing the roles of the sets we get a contradicition also in the case when there is $\mu \in F(A) \setminus F(L(A))$. ∎

For a real linear operator $L : V \to V$ define $\overline{L} : V \to V$ by $\overline{L}(A) = L(\overline{A})$ for all $A \in V$. With this notation, we are now ready to present our main result.

Theorem 5 (a) *A real linear map $L : T(n_1, \ldots, n_k) \to T(n_1, \ldots, n_k)$ is a numerical radius isometry if and only if there exists a complex unit ξ such that ξL or $\xi \overline{L}$ preserves the numerical range.*
(b) *A real linear map $L : D(n_1, \ldots, n_k) \to D(n_1, \ldots, n_k)$ is a numerical radius isometry if and only if there exists a mapping $S : D(n_1, \ldots, n_k) \to D(n_1, \ldots, n_k)$ defined by*

$$A_1 \oplus \cdots \oplus A_k \mapsto S_1(A_1) \oplus \cdots \oplus S_k(A_k)$$

such that $S \circ L$ preserves the numerical range, where $S_j : M_{n_j} \to M_{n_j}$ has the form

$$A_j \mapsto \mu_j A_j, \qquad or \qquad A_j \mapsto \mu_j \overline{A_j}$$

for some complex unit μ_j.

Proof. The ($\Leftarrow$) is clear for both (a) and (b). For the converse, suppose L is a real linear numerical radius isometry on $T(n_1, \ldots, n_k)$. By Corollary 3, there exists a complex unit μ such that $(L(I), L(iI)) = \mu(I, \pm iI)$. Thus, $\bar{\mu} L$ or $\bar{\mu} \overline{L}$ satisfy the hypothesis of Lemma 4 with $F(A) = W(A)$. The result follows. Suppose L is a real linear numerical radius isometry on $D(n_1, \ldots, n_k)$. By Corollary 3, $(L(I), L(iI)) = (\mu_1 I_{n_1} \oplus \cdots \oplus \mu_k I_{n_k}, d_1 \mu_1 I_{n_1} \oplus \cdots \oplus d_k \mu_k I_{n_k})$. So, there exists a mapping of the form S such that $S \circ L$ satisfy the hypothesis of Lemma 4 with $F(A) = W(A)$. The result follows. ∎

In [3], the author characterized additive maps which preserve the numerical range on different types of matrix algebras. We can use the results in [3] to give explicit description of real linear numerical radius isometries on $V = D(n_1, \ldots, n_k)$ or $T(n_1, \ldots, n_k)$. For easy reference, we list the results explicitly in the following. Interested readers can read [3] for details. For simplicity and clarity, we present the results for M_n, $D(n_1, \ldots, n_k)$, $T_n = T(1, \ldots, 1)$ - the algebra of upper triangular matrices in M_n, and $T(n_1, \ldots, n_k)$, in separate corollaries.

By Theorem 5 (b) and [3, Corollary], we have the following.

Corollary 6 *Suppose L is a real linear numerical radius isometry on M_n. Then there exist a complex unit ξ and unitary U such that L is of one of the forms:*

$$A \mapsto \xi U^* A U, \ A \mapsto \xi U^* A^t U, \ A \mapsto \xi U^* \overline{A} U, \ A \mapsto \xi U^* A^* U. \tag{2}$$

It is interesting to note that the first two types of mappings in 2 are complex linear numerical radius isometries (see [4]), and the last two types are just a composition of the first two types with complex conjugation, i.e., the mapping $A \mapsto \overline{A}$.

By Theorem 5 (b) and [3, Theorem 8], we have the following.

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Corollary 7 *Suppose L is a real linear numerical radius isometry on $D(n_1, \ldots, n_k)$. Then there exists a permutation $(j_1, \ldots, j_k)$ of $(1, \ldots, k)$ satisfying $n_t = n_{j_t}$ for $t = 1, \ldots, k$, such that L has the form*

$$A_1 \oplus \cdots \oplus A_k \mapsto L_1(A_{j_1}) \oplus \cdots \oplus L_k(A_{j_k}),$$

where each $L_j : M_{n_j} \to M_{n_j}$ has one of the forms in (2) for some unitary $U \in M_{n_j}$.

The structure of numerical range preservers on T_n is more complicated. Lešnjak [3] showed that the set of additive numerical range preservers on T_n is a group generated by operators of the forms:

(i) $A \mapsto DAD^*$ for some diagonal unitary matrix $D \in M_n$,

(ii) $A \mapsto EA^tE$ with $E = \sum_{j+k=n+1} E_{jk} \in M_n$, and

(iii) $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & A_{11}^t & \\ 0 & & & \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \bar{a}_{1n} & \cdots & \bar{a}_{12} \\ 0 & & & \\ \vdots & & \tilde{E}A_{11}\tilde{E} & \\ 0 & & & \end{pmatrix}$ with $\tilde{E} = \sum_{j+k=n} E_{jk} \in M_{n-1}$.

By Theorem 5 (a) and [3, Theorem 11], we have the following.

Corollary 8 *The set of real linear numerical radius isometries on T_n is a group generated by operators of the forms (i) – (iii),*

(iv) $A \mapsto \mu A$ *for some complex unit μ, and*

(v) $A \mapsto \overline{A}$.

Note that mappings of the form (i), (ii) and (iv) generate the group of complex linear isometries for the numerical radius. The mapping in (v) is a natural addition to the group of real linear isometries. It is surprising and interesting to see that the mapping in (iii) also preserves the numerical radius, and it is the only additional mapping needed to generate the isometry group.

The description of the group of real linear isometries on $T(n_1, \ldots, n_k)$ is more delicate. First, one may relax the mapping in (i) to the following:

(i′) $A \mapsto U^*AU$ for some unitary $U \in T(n_1, \ldots, n_k)$.

Next, the mapping in (ii) does not always work on $T(n_1, \ldots, n_k)$ without additional assumptions on $n_1, \ldots, n_k$:

(ii′) $A \mapsto EA^tE$ with $E = \sum_{j+k=n+1} E_{jk} \in M_n$ provided $n_j = n_{k+1-j}$ for $j = 1, \ldots, k$.

Similarly, a mapping of the form (iii) does not work on $T(n_1, \ldots, n_k)$ without additional assumption on $n_1, \ldots, n_k$ and suitable modification:

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(iii') there exists $1 \leq r < k$ such that $n_j = n_{r+1-j}$ for $j = 1, \ldots, r$, and $n_{r+j} = n_{r+1-j}$ for $j = 1, \ldots, k - r$, so that one can consider the mapping on $T(n_1, \ldots, n_k)$ defined by

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \mapsto \begin{pmatrix} E_1 X^t E_1 & E_1 \overline{Y} E_2 \\ 0 & E_2 Z^t E_2 \end{pmatrix},$$

where $X \in T(n_1, \ldots, n_r)$, $Z \in T(n_{r+1}, \ldots, n_k)$, E_1 and E_2 are square zero-one matrices of appropriate sizes having ones on the anti-diagonals.

By Theorem 5 (a) and [3, Theorem 18], we have the following.

Corollary 9 *The set of real linear numerical radius isometries on $T(n_1, \ldots, n_k)$ is a group generated by operators of the forms* (i'), (ii'), (iii'), (iv) *and* (v).

We note that our investigation is just a special case of a more general problem, namely, studying mappings f on a normed vector space V that satisfy $\|f(X) - f(Y)\| = \|X - Y\|$ for all $X, Y \in V$. If V is finite dimensional, then we can assume with no loss of generality that $f(0) = 0$, and thus f is real linear. If V is a complex linear space, it is natural to ask whether real linear isometries are either complex linear isometries or complex linear isometries composed by the complex conjugation. Our results show that this may or may not be true. It is interesting to identify those complex normed spaces having this property.

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