Determinantal representations of hyperbolic forms via weighted shift matrices

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Based on joint work with Hiroshi Nakazato

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1. Introduction

2. Curves

3. Cyclic weighted shifts

4. Cubic forms

5. Quartic and quintic forms
1. Introduction

Let $A$ be an $n \times n$ matrix. The numerical range of $A$ is defined as

$$W(A) = \{ x^* Ax; x \in \mathbb{C}^n, \|x\| = 1 \}.$$

Rudolf Kippenhahn (1951) characterized that $W(A)$ is the convex hull of the real affine part of the dual curve of the curve $F_A(t, x, y) = 0$, where the associated homogeneous polynomial

$$F_A(t, x, y) = \det \left( tl_n + x \Re(A) + y \Im(A) \right),$$

and $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$. 
1. Introduction

\[ A = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & -1 & 2 & 0 \\
0 & -2 & 2 & 1 \\
0 & 0 & -1 & 3
\end{bmatrix}. \]

Then

\[ F_A(t, x, y) = \det \left( tl_n + x \Re(A) + y \Im(A) \right), \]

\[ = t^4 + 3t^3x - 3t^2x^2 - 6t^2y^2 - 7tx^3 - 11txy^2 + 6x^4 + 5x^2y^2 + y^4. \]

\[ F_A(1, x, y) = 0 \quad \text{Dual curve} \]
Question

Let \( p(x_1, x_2, \ldots, x_m) \) be a homogeneous polynomial of degree \( n \) in \( m \) variables.
Do there exist \( n \times n \) matrices \( A_1, A_2, \ldots, A_m \) so that \( p(x_1, x_2, \ldots, x_m) = \det(x_1 A_1 + x_2 A_2 + \cdots + x_m A_m) \)?
1. Introduction

**Question**

Let \( p(x_1, x_2, \ldots, x_m) \) be a homogeneous polynomial of degree \( n \) in \( m \) variables. Do there exist \( n \times n \) matrices \( A_1, A_2, \ldots, A_m \) so that \( p(x_1, x_2, \ldots, x_m) = \det(x_1 A_1 + x_2 A_2 + \cdots + x_m A_m) \)?

In other words,

Which surfaces of \( \mathbb{R}^m \) can be represented with Linear Matrix Determinants?
1. Introduction

A. C. Dixon, 1901

If $m = 3$, there exist complex symmetric matrices $A_1, A_2, A_3$ such that $p(x_1, x_2, x_3) = \det(x_1 A_1 + x_2 A_2 + x_3 A_3)$. 

1. Introduction

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Leonard Eugene Dickson, 1921

For any homogeneous polynomial \( p(x_1, x_2, \ldots, x_m) \) of degree \( n \) in \( m \) variables, there exist \( n \times n \) \textbf{complex matrices} \( A_1, A_2, \ldots, A_m \) satisfying \( p(x_1, x_2, \ldots, x_m) = \det(x_1 A_1 + x_2 A_2 + \cdots + x_m A_m) \) if and only if \( m = 3, n \text{ any} \), \textbf{or} \( m = 4, n = 2, 3 \).
Lax-Fiedler conjecture

Peter David Lax, 1958, conjectured that if $F(t, x, y)$ is a real hyperbolic ternary form of degree $n$ with $F(1, 0, 0) = 1$, then there exist $n \times n$ real symmetric matrices $H$ and $K$ satisfying

$$F(t, x, y) = \det(tI_n + xH + yK) = F_{H+iK}(t, x, y).$$

Compare: $F(t, x, y) = \det(tA_1 + xA_2 + yA_3)$.

- $F(t, x, y)$ is hyperbolic w.r.t $(1, 0, 0)$: $F(t, x, y) = 0$ has only real roots in $t$ for any real numbers $x, y$, and $F(1, 0, 0) \neq 0$. 
1. Introduction

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Peter David Lax, 1958, conjectured that if \( F(t, x, y) \) is a real hyperbolic ternary form of degree \( n \) with \( F(1, 0, 0) = 1 \), then there exist \( n \times n \) real symmetric matrices \( H \) and \( K \) satisfying

\[
F(t, x, y) = \det(tI_n + xH + yK) \quad (= F_{H+iK}(t, x, y)).
\]

Compare: \( F(t, x, y) = \det(tA_1 + xA_2 + yA_3) \).

- \( F(t, x, y) \) is hyperbolic w.r.t \( (1, 0, 0) \): \( F(t, x, y) = 0 \) has only real roots in \( t \) for any real numbers \( x, y \), and \( F(1, 0, 0) \neq 0 \).

Miroslav Fiedler, 1981, conjectured the existence of the real symmetric matrices \( H, K \) relaxing to complex hermitian matrices.
1. Introduction

**Difference**

\[ F(t, x, y) = \det(tA_1 + xA_2 + yA_3) \] and
\[ F(t, x, y) = \det(tI_n + xH + yK). \]

Consider \( p(t, x, y) = -t^2 + x^2 - y^2. \)

Then \( F(t, x, y) = \det(tA_1 + xA_2 + yA_3), \) where

\[ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

But

\[ F(t, x, y) \neq \det(tI_2 + xH + yK). \]
1. Introduction

Answer to Lax-Fiedler conjecture

J. William Helton and Victor Vinnikov, 2007, confirmed the Lax conjecture is true. Namely, if \( F(t, x, y) \) is a real hyperbolic ternary form of degree \( n \) with \( F(1, 0, 0) = 1 \), then there exist \( n \times n \) real symmetric matrices \( H \) and \( K \) satisfying

\[
F(t, x, y) = \det(t l_n + xH + yK).
\]
2. Curves

Consider a roulette curve

\[ z(\theta) = \exp(-i(m - 1)\theta) + a \exp(im\theta), \quad a > 1. \]

\[ x(\theta) = \cos((m - 1)\theta) + a \cos(m\theta), \quad y(\theta) = -\sin((m - 1)\theta) + a \sin(m\theta). \]
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**Theorem 2.1** The roulette curve is the orbit of a point mass under a central force, and there exists a \( 2m \times 2m \) matrix \( A \) such that

\[ F_A(1, x(\theta), y(\theta)) = \det(I_{2m} + x(\theta) \Re(A) + y(\theta) \Im(A)) = 0. \]
2. Curves

\[ z(\theta) = e^{-i(m-1)\theta} + ae^{im\theta}, \ a > 1. \]

\[ m = 3, \ a = 4 \]

Dual curve
3. Cyclic weighted shifts

For arbitrary complex numbers $a_1, a_2, \ldots, a_n$, we consider an $n \times n$ cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$ defined as

$$S = S(a_1, a_2, \ldots, a_n) = \begin{pmatrix}
0 & a_1 & 0 & 0 & \ldots & 0 \\
0 & 0 & a_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & \vdots & \vdots & \ddots & a_{n-1} \\
ap_n & 0 & \ldots & \ldots & \ldots & 0
\end{pmatrix}.$$

- M. C. Tsai, P. Y. Wu, 2011, flat portion
- H. L. Gau, M. C. Tsai and H. C. Wang, 2013, reducibility
- M. C. Tsai, H. L. Gau and H. C. Wang, 2014, flat portion
3. Cyclic weighted shifts

Let $S = S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero real weights, $F_S(t, x, y) = \det(tI_n + x \Re(S) + y \Im(S))$.

(i) $F_S(t, x, y)$ is hyperbolic w.r.t. $(1, 0, 0)$ and $F(1, 0, 0) = 1$. 

Let \( S = S(a_1, a_2, \ldots, a_n) \) be a cyclic weighted shift matrix with non-zero real weights, \( F_S(t, x, y) = \det(tI_n + x\Re(S) + y\Im(S)) \).

(i) \( F_S(t, x, y) \) is hyperbolic w.r.t. \((1, 0, 0)\) and \( F(1, 0, 0) = 1 \).

(ii) \( F_S(t, x, y) \) is weakly circular symmetric:

\[
F_S(t, x, y) = F_S(t, \cos(2\pi/n)x - \sin(2\pi/n)y, \sin(2\pi/n)x + \cos(2\pi/n)y).
\]

\( S \) is unitarily similar to \( e^{-2\pi i/n} S \).

\[
e^{-2\pi i/n} S = (\cos(2\pi/n)\Re(S) + \sin(2\pi/n)\Im(S)) + i(-\sin(2\pi/n)\Re(S) + \cos(2\pi/n)\Im(S)).
\]
3. Cyclic weighted shifts

\[(iii) \quad F_S(t, x, -y) = F_S(t, x, y).\]
3. Cyclic weighted shifts

(iii) \( F_S(t, x, -y) = F_S(t, x, y) \).

(iv) \( F_S(t, -1, -i) = t^n - a \) for some nonzero real number \( a \).
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- \( n = 2p + 1 \),
  \[
  F_S(t, \cos \theta, \sin \theta) = t^n + \gamma_1 t^{n-2} + \gamma_2 t^{n-4} + \cdots + \gamma_p t + \gamma_{p+1} \cos(n\theta).
  \]
3. Cyclic weighted shifts

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- \( n = 2p \),
  \( F_S(t, \cos \theta, \sin \theta) = t^n + \gamma_1 t^{n-2} + \gamma_2 t^{n-4} + \cdots + \gamma_p + \gamma_{p+1} \cos(n\theta) \).
3. Cyclic weighted shifts

Inverse problem

Let $F(t, x, y)$ be a ternary form of degree $n$ satisfying conditions $(i)-(iv)$:

$(i)$ $F(t, x, y)$ is hyperbolic w.r.t. $(1, 0, 0)$ and $F(1, 0, 0) = 1$.

$(ii)$ $F(t, x, y)$ is weakly circular symmetric:

$$F(t, x, y) = F(t, \cos(2\pi/n)x - \sin(2\pi/n)y, \sin(2\pi/n)x + \cos(2\pi/n)y).$$

$(iii)$ $F(t, x, -y) = F(t, x, y)$.

$(iv)$ $F(t, -1, -i) = t^n - a$ for some positive number $a$.

Does there exist a cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$ so that $F(t, x, y) = F_S(t, x, y)$?
4. Cubic forms

\( n = 3 \)

**Theorem 4.1** Let \( A \) be a \( 3 \times 3 \) complex matrix and its ternary form \( F_A(t, x, y) \) satisfy conditions (i), (ii), (iii), (iv). Then the matrix \( A \) is unitarily equivalent to a cyclic weighted shift matrix \( S(a_1, a_2, a_3) \) with positive weights.

- \( F_A(t, x, y) = t^3 - \eta_1 t(x^2 + y^2) + \eta_2(x^3 - 3xy^2). \)
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- \( F_A(t, x, y) = t^3 - \eta_1 t(x^2 + y^2) + \eta_2(x^3 - 3xy^2). \)
- \( \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta. \)
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- \( F_A(t, x, y) = t^3 - \eta_1 t(x^2 + y^2) + \eta_2(x^3 - 3xy^2). \)
- \( \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta. \)
- \( F_A(t, \cos \theta, \sin \theta) = t^3 - \eta_1 t + \eta_2 \cos(3\theta) = 0 \) in \( t \) has 3 real roots for any \( 0 \leq \theta \leq 2\pi. \)
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- $F_A(t, x, y) = t^3 - \eta_1 t(x^2 + y^2) + \eta_2(x^3 - 3xy^2)$.
- $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$.
- $F_A(t, \cos \theta, \sin \theta) = t^3 - \eta_1 t + \eta_2 \cos(3\theta) = 0$ in $t$ has 3 real roots for any $0 \leq \theta \leq 2\pi$.
- Taking $\theta = \pi/6$ and $\theta = 0$, we find that $\eta_1 > 0$ and $4\eta_1^3 - 27\eta_2^2 \geq 0$. 
$n = 3$

**Theorem 4.2** Let $F(t, x, y)$ be a real cubic form satisfying the conditions (i), (ii), (iii), (iv). Then $F(t, x, y) = F_S(t, x, y)$ for some cyclic weighted shift matrix $S = S(a_1, a_2, a_1)$ with positive weights $a_1, a_2, a_1$.

- $4F(t, x, y) = 4t^3 - 3\gamma_1 t(x^2 + y^2) + \gamma_2(x^3 - 3xy^2)$. 
4. Cubic forms

\( n = 3 \)

**Theorem 4.2** Let \( F(t, x, y) \) be a real cubic form satisfying the conditions \((i), (ii), (iii), (iv)\). Then \( F(t, x, y) = F_S(t, x, y) \) for some cyclic weighted shift matrix \( S = S(a_1, a_2, a_1) \) with positive weights \( a_1, a_2, a_1 \).

- \( 4F(t, x, y) = 4t^3 - 3\gamma_1 t(x^2 + y^2) + \gamma_2(x^3 - 3xy^2) \).
- \( 4F_{S(a_1,a_2,a_1)}(t, x, y) = 4t^3 - (2a_1^2 + a_2^2)t(x^2 + y^2) + a_1^2a_2(x^3 - 3xy^2) \).
4. Cubic forms

It is obvious that two unitarily equivalent matrices $A$ and $B$ have the same associated forms $F_A(t, x, y) = F_B(t, x, y)$, but the converse is not true even for the cyclic weighted shift matrices. Consider

$$A = \begin{pmatrix} 0 & 6 & 0 \\ 0 & 0 & 8 \\ 7 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ a & 0 & 0 \end{pmatrix},$$

where $a$ and $b$ are positive roots of $a^2b = 336$, $2a^2 + b^2 = 149$. The numerical solutions are $a = 7.57091$ and $b = 5.86196$. Then

$$F_A(t, x, y) = F_B(t, x, y) = \frac{1}{4}(4t^3 - 149tx^2 + 336x^3 - 149ty^2 - 1008xy^2).$$

But, $A$ and $B$ are not unitarily equivalent.
5. Quartic and quintic forms

We give an example to show that the result of Theorem 4.1 may fail for $4 \times 4$ matrices. Consider the matrix

$$A = \begin{pmatrix}
1 & \frac{-1+i}{8} & \frac{-2}{5} & \frac{-1-i}{140}(\frac{3}{7})^{1/2} \\
0 & i & \frac{1+i}{5}(\frac{3}{7})^{1/2} & \frac{-2i}{5} \\
0 & 0 & -1 & \frac{-1-i}{140}(\frac{3}{7})^{1/2} \\
0 & 0 & 0 & -i
\end{pmatrix}.$$ 

Then

$$716800 F_A(t, x, y) = 716800t^4 - 798720t^2(x^2 + y^2) + 30303(x^4 + y^4) + 777406x^2y^2.$$ 

The form $F_A(t, x, y)$ satisfies the conditions $(i)$-$(iv)$, but $A$ is not unitarily equivalent to a cyclic weighted shift matrix.
5. Quartic and quintic forms

\[ n(n - 3)/2 \] singular points

Nakazato-Chien, 2013

**Theorem A.** Let \( S = S(a_1, a_2, \ldots, a_n), \ n \geq 3, \) be a unitarily irreducible cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the associative curve \( F_S(t, x, y) = 0 \) is at most \( n(n - 3)/2. \)

H. L. Gau, M. C. Tsai, H. C. Wang, 2013

**Theorem B.** A cyclic weighted shift matrix \( S(a_1, a_2, \ldots, a_n) \) is unitarily irreducible if and only if its weights \( a_1, a_2, \ldots, a_n \) are non-periodic.
5. Quartic and quintic forms

\( n = 4, \ n(n - 3)/2 \) singular points

**Theorem 5.1** Let \( F(t, x, y) \) be a real quartic ternary form satisfying the conditions \((i), (ii), (iii), (iv)\). Assume that the curve \( F(t, x, y) = 0 \) has two real singular points on the line \( t = 0 \) and has no other singular points. Then \( F(t, x, y) = F_S(t, x, y) \) for some cyclic weighted shift matrix \( S = S(a_1, a_2, a_2, a_1) \) with positive weights.

- \( F(t, x, y) = t^4 - c_1 t^2 (x^2 + y^2) + c_2 (x^2 + y^2)^2 - c_3 (x^4 - 6x^2 y^2 + y^4). \)
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- \( F(t, x, y) = t^4 - c_1 t^2(x^2 + y^2) + c_2(x^2 + y^2)^2 - c_3(x^4 - 6x^2 y^2 + y^4) \).
- \( \cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta \).
5. Quartic and quintic forms

\( n = 4, \quad n(n - 3)/2 \) singular points

**Theorem 5.1** Let \( F(t, x, y) \) be a real quartic ternary form satisfying the conditions (i), (ii), (iii), (iv). Assume that the curve \( F(t, x, y) = 0 \) has two real singular points on the line \( t = 0 \) and has no other singular points. Then \( F(t, x, y) = F_S(t, x, y) \) for some cyclic weighted shift matrix \( S = S(a_1, a_2, a_2, a_1) \) with positive weights.

- \( F(t, x, y) = t^4 - c_1 t^2(x^2 + y^2) + c_2(x^2 + y^2)^2 - c_3(x^4 - 6x^2y^2 + y^4). \)
- \( \cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta. \)
- \( F_S(a_1, a_1, a_2, a_2)(t, x, y) = t^4 - \frac{1}{2}(a_1^2 + a_2^2)t^2(x^2 + y^2) + a_1^2 a_2^2 x^2 y^2. \)
5. Quartic and quintic forms

\[ n = 5, \ n(n - 3)/2 \text{ singular points} \]

**Theorem 5.2** Let \( F(t, x, y) \) be a real quintic ternary form satisfying the conditions \((i)-(iv)\). Assume the singular points of the curve \( F(t, x, y) = 0 \) are
\( (1, -\cos(2k\pi/5), \sin(2k\pi/5)), \ k = 0, 1, 2, 3, 4. \) Then
\( F(t, x, y) = F_S(t, x, y) \) for some cyclic weighted shift matrix \( S = S(a_1, a_2, a_3, a_2, a_1) \) with real reversible non-zero weights.

- \( F(t, x, y) = t^5 + c_1 t^3(x^2 + y^2) + c_2 t(x^2 + y^2)^2 + c_3(x^5 - 10x^3y^2 + 5xy^4). \)
5. Quartic and quintic forms

\( n = 5, \quad n(n - 3)/2 \) singular points

**Theorem 5.2** Let \( F(t, x, y) \) be a real quintic ternary form satisfying the conditions \((i)-(iv)\). Assume the singular points of the curve \( F(t, x, y) = 0 \) are
\( (1, -\cos(2k\pi/5), \sin(2k\pi/5)), \quad k = 0, 1, 2, 3, 4. \)
Then \( F(t, x, y) = F_S(t, x, y) \) for some cyclic weighted shift matrix \( S = S(a_1, a_2, a_3, a_2, a_1) \) with real reversible non-zero weights.

- \( F(t, x, y) = t^5 + c_1 t^3(x^2 + y^2) + c_2 t(x^2 + y^2)^2 + c_3(x^5 - 10x^3y^2 + 5xy^4). \)
- \( \cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta. \)
Unsolved

This determinantal method can be applied to the cases $n = 6, 7, 8$. 
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However, we can not efficiently perform this method for $n \geq 9$ because of the complexity on configuration of the singular points of the algebraic curve $F(t, x, y) = 0$, 

Unsolved

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However, we can not efficiently perform this method for $n \geq 9$ because of the complexity on configuration of the singular points of the algebraic curve $F(t, x, y) = 0$.

Let $F(t, x, y)$ be a real ternary form of degree $n$ satisfying the conditions $(i), (ii), (iii), (iv)$. Assume that the curve $F(t, x, y) = 0$ has $n(n - 3)/2$ real singular points. Then $F(t, x, y) = FS(t, x, y)$ for some cyclic weighted shift matrix $S = S(a_1, a_2, a_3, \ldots, a_3, a_2, a_1)$ with positive weights.
Thank you for your attention!