FACTORING A QUADRATIC OPERATOR AS A PRODUCT OF TWO POSITIVE CONTRACTIONS

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Abstract. Let $T$ be a quadratic operator on a complex Hilbert space $H$. We show that $T$ can be written as a product of two positive contractions if and only if $T$ is of the form

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix}$$

on $H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$

for some $a, b \in [0, 1]$ and strictly positive operator $P$ with $\|P\| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}$. Also, we give a necessary condition for a bounded linear operator $T$ with operator matrix $\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$ on $H \oplus K$ that can be written as a product of two positive contractions.

1. Introduction

There has been considerable interest in studying factorization of bounded linear operators (see [2, 3, 4, 5, 15]). For example, a $2 \times 2$ matrix $C$ can be written as a product of two orthogonal projections if and only if $C$ is the identity operator or $C$ is unitarily similar to

$$\begin{pmatrix} a & \sqrt{a(1-a)} \\ 0 & 0 \end{pmatrix}$$

for some $a \in [0, 1]$. For more results about products of orthogonal projections, one may consult [1, 7, 8, 11]. Note that one can write an $n \times n$ matrix $C$ as a product of two positive (semi-definite) operators exactly when $C$ is similar to a positive operator (see [14, Theorem 2.2]). However, in the infinite dimensional case, the product of two positive operators may not be similar to a positive operator (see [12], [15, Example 2.11]). For more development in this direction, one may consult [12, 14, 15].

In this paper, we study the problem when a bounded linear operator $T$ on a complex Hilbert space $H$ can be written as a product of two positive contractions. In this case, $T$ must be a contraction, and we have that

$$-I/8 \leq \text{Re} \, T \quad \text{and} \quad -I/4 \leq \text{Im} \, T \leq I/4$$

(see [10, Theorem 1.1 and Corollary 4.3]). In Proposition 2.4, we give a necessary condition for this problem when $T$ has operator matrix

$$\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$$
on $H \oplus K$.

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In such a case, $T_1$ and $T_2$ must also be products of two positive contractions. This is an extension of the result of Wu in [14, Corollary 2.3] concerning the finite dimensional case. However, even for a $2 \times 2$ matrix $C$, it is not easy to determine when it is the product of two positive contractions. For example, consider

$$C = \frac{1}{25} \begin{pmatrix} 9 & 3 \\ 0 & 16 \end{pmatrix}.$$ 

The diagonalizable contraction $C$ is similar to a positive operator. Thus it is a product of two positive operators. Moreover, $C$ satisfies $-I/8 \leq \text{Re} C$ and $-I/4 \leq \text{Im} C \leq I/4$. However, we will see that $C$ cannot be written as a product of two positive contractions by Lemma 2.1.

Let $B(H)$ be the algebra of bounded linear operators acting on a complex Hilbert space $H$. We identify $B(H)$ with $M_n$, the algebra of $n \times n$ complex matrices, if $H$ has finite dimension $n$. Recall that a bounded linear operator $T \in B(H)$ is positive (resp., strictly positive) if $\langle Th, h \rangle \geq 0$ (resp., $\langle Th, h \rangle > 0$) for every $h \neq 0$ in $H$. We write as usual $T \geq 0$ (resp., $T > 0$) when $T$ is positive (resp., strictly positive).

We call $T \in B(H)$ a quadratic operator if $(T - aI)(T - bI) = 0$ for some scalars $a, b \in \mathbb{C}$. Every quadratic operator $T \in B(H)$ is unitarily similar to $aI \oplus bI \oplus (aI P \oplus bI)$ on $H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$ for some $a, b \in \mathbb{C}$ and $P > 0$ (see [13]). In this paper, we prove the following.

**Theorem 1.1.** A quadratic operator $T \in B(H)$ with operator matrix

$$aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \text{ on } H_1 \oplus H_2 \oplus (H_3 \oplus H_3)$$

for some $a, b \in \mathbb{C}$ and $P > 0$, can be written as a product of two positive contractions if and only if

$$a, b \in [0, 1], \text{ and } \|P\| \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1 - a)(1 - b)}.$$ 

2. **Proof**

First we consider the $2 \times 2$ case so that we can identify $B(H) = M_2$ and $H = \mathbb{C}^2$.

**Lemma 2.1.** Suppose $C = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}$ with $z \geq 0$. Then $C$ is a product of two positive contractions if and only if $a, b \in [0, 1]$,

$$z \in S = \{c : 0 \leq c \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1 - a)(1 - b)}\}.$$ 

If the above equivalent conditions hold, then there are continuous maps $a_{ij}(z), b_{ij}(z)$ for $1 \leq i, j \leq 2$ with

\[
\begin{aligned}
0 &\leq a_{ii}(z), \quad b_{ii}(z) \leq 1, \quad a_{12}(z) = a_{21}(z) \geq 0, \quad b_{12}(z) = b_{21}(z) \leq 0, \\
0 &\leq (a_{ij}(z)) \leq I, \quad 0 \leq (b_{ij}(z)) \leq I.
\end{aligned}
\]
such that

\[(2.2) \quad (a_{ij}(z))(b_{ij}(z)) = \begin{pmatrix} a & z \\ 0 & b \end{pmatrix}, \quad z \in S.\]

**Proof.** We first prove the sufficiency. Without loss of generality, we may assume \(0 \leq a \leq b \leq 1.\) If \(a = b\) or \(b = 1,\) then \(z = 0\) and \(C = \text{diag}(a,1)\text{diag}(1,b).\) In the following, we may assume \(0 \leq a < b < 1,\) and consider two cases.

**Case 1.** \(0 = a < b < 1.\) For \(z \in S,\) we have that \(z^2 \leq b(1-b)\) and hence \((z^2/b) + b \leq (1-b) + b = 1.\)

Consider

\[
A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \begin{pmatrix} z^2/b & z \\ z & b \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

Then \(A\) is rank 1 with eigenvalue \((z^2/b) + b,\) and \(C = AB.\) Evidently, \(a_{ij}(z), b_{ij}(z)\) are continuous maps for \(1 \leq i,j \leq 2\) and satisfy (2.1), (2.2).

**Case 2.** \(0 < a < b < 1.\) For \(z \in S,\) we have

\[
a + b - \frac{z^2}{(1-a)(1-b)} \geq a + b - (\sqrt{a} - \sqrt{b})^2 = 2\sqrt{ab}.
\]

Let \(\lambda_1(z) \geq \lambda_2(z)\) be roots of the equation

\[
\lambda^2 - (a + b - \frac{z^2}{(1-a)(1-b)})\lambda + ab = 0.
\]

Then, \(a \leq \lambda_2(z) \leq \lambda_1(z) \leq b\) and \(\lambda_1(z), \lambda_2(z)\) are continuous maps on \(z \in S.\) Note that

\[
\lambda_1(z)\lambda_2(z) = ab, \quad \lambda_1(z) + \lambda_2(z) = a + b - \frac{z^2}{(1-a)(1-b)}.
\]

We have

\[(2.3) \quad z = \sqrt{\frac{(1-a)(1-b)(\lambda_j - a)(b - \lambda_j)}{\lambda_j}}, \quad j = 1,2.\]

We will construct

\[
A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} = \gamma \begin{pmatrix} a_3 & -a_2 \\ -a_2 & a_4 \end{pmatrix}
\]

such that \(A\) has eigenvalues 1, \(\lambda_1,\) \(B\) has eigenvalues 1, \(\lambda_2,\) and \(C = AB.\) First, we set

\[(2.4) \quad \gamma = \frac{\lambda_2}{b} = \frac{a}{\lambda_1} < 1.
\]

Because \(1 - b - \gamma + b\gamma = (1-b)(1-\gamma) > 0,\) we can let

\[
a_3 = \frac{b - a}{1 + b\gamma - \gamma - a} < \frac{b - a}{b - a} = 1
\]
so that by (2.4),
\[
a_3 - \lambda_1 = \frac{(b - a)}{(1 + b\gamma - \gamma - a)} - \frac{a}{\gamma} = \frac{\gamma b - \gamma a - a - \gamma ab + \gamma a + a^2}{\gamma(1 + b\gamma - \gamma - a)}
\]
\[
= \frac{1}{2} \frac{(\gamma b - a)(1 - a)}{(1 + b\gamma - \gamma - a)} = \frac{(b - \lambda_1)(1 - a)}{(1 + b\gamma - \gamma - a)} \geq 0.
\]
Then we can let
\[
a_1 = 1 + \lambda_1 - a_3 > 0 \quad \text{so that} \quad a_1 + a_3 = 1 + \lambda_1
\]
and
\[
a_2 = \sqrt{a_1a_3 - \lambda_1} = \sqrt{(1 + \lambda_1 - a_3)a_3 - \lambda_1} = \sqrt{(1 - a_3)(a_3 - \lambda_1)} \quad \text{so that} \quad a_1a_3 - a_2^2 = \lambda_1.
\]
As a result, \(a_1 + a_3 = 1 + \lambda_1\), \(\det(A) = \lambda_1\), and hence \(A\) has eigenvalues \(1, \lambda_1\). Further, let
\[
a_4 = \frac{1}{a_3} \left( \frac{\lambda_2}{\gamma^2} + a_2^2 \right) \quad \text{so that} \quad \gamma^2(a_3a_4 - a_2^2) = \lambda_2.
\]
Then by (2.4),
\[
\gamma(a_3 + a_4) = \gamma a_3 + \frac{\gamma}{a_3} \left( \frac{\lambda_2}{\gamma^2} + a_2^2 \right) = \frac{\gamma}{a_3} \left( \frac{\lambda_2}{\gamma^2} + (a_3 - \lambda + \lambda_1a_3) \right)
\]
\[
= \frac{\gamma}{a_3} \left( \frac{\lambda_2}{\gamma^2} - \lambda_1 \right) + \gamma(1 + \lambda_1) = \frac{\gamma(b - a)}{\gamma} + \gamma(1 + \lambda_1) = 1 + b\gamma - \gamma - a + \gamma + \gamma\lambda_1 = 1 + \lambda_2.
\]
As a result, \(\text{tr}B = 1 + \lambda_2\) and \(\det(B) = \lambda_2\). Therefore, \(B\) has eigenvalues \(1, \lambda_2\). Denote by \((AB)_{ij}\) the \((i, j)\) entry of \(AB\). By (2.4),
\[
(AB)_{11} = \gamma(a_1a_3 - a_2^2) = \gamma\lambda_1 = a, \quad (AB)_{22} = \gamma(a_3a_4 - a_2^2) = \gamma(\lambda_2/\gamma^2) = b.
\]
Clearly, \((AB)_{21} = \gamma(a_2a_3 - a_3a_2) = 0\). By (2.4) and (2.3),
\[
(AB)_{12} = \gamma a_2(a_4 - a_1) = \gamma \sqrt{(1 - a_3)(a_3 - \lambda_1)((a_3 + a_4) - (a_3 + a_1))}
\]
\[
= \gamma \sqrt{(1 - b - \gamma + b\gamma)(b - \lambda_1)(1 - a)} \left( \frac{(1 + \lambda_2)}{\gamma} - (1 + \lambda_1) \right)
\]
\[
= \sqrt{(1 - b)(1 - \gamma)(1 - a)(b - \lambda_1)} \left(1 + \lambda_2 - \gamma - \lambda_1 \right)
\]
\[
= \sqrt{(1 - b)(1 - a)(1 - \gamma)(b - \lambda_1)} = \sqrt{(1 - b)(1 - a)(\lambda_1 - a)(b - \lambda_1)} = z.
\]

For the converse, since \(A, B\) are positive contractions with \(\sigma(C) = \sigma(AB) = \sigma(B^{1/2}AB^{1/2}) \subseteq [0, \infty)\), we have \(0 \leq a, b \leq 1\). Without loss of generality, we may assume \(a \leq b\). First, consider \(\|A\| = \|B\| = 1\). Then the assumption \(C = AB\) implies \(C\) is unitarily similar to
\[
\begin{pmatrix}
\alpha_1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
b_1 & b_2 \\
b_2 & b_4
\end{pmatrix} = \begin{pmatrix}
\alpha_1b_1 & \alpha_1b_2 \\
b_2 & b_4
\end{pmatrix}.
\]
where \( \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} \) is unitarily similar to \( \begin{pmatrix} \alpha_2 & 0 \\ 0 & 1 \end{pmatrix} \) for some \( 0 \leq \alpha_1, \alpha_2 \leq 1 \), \( \alpha_2 \leq b_1, b_4 \leq 1 \) and \( b_2 \geq 0 \). Thus we have \( 1 + \alpha_2 = b_1 + b_4 \), \( a + b = \alpha_1 b_1 + b_4 \), \( ab = \alpha_1 \alpha_2 = \alpha_1 (b_1 b_4 - b_2) \), and \( a^2 + b^2 + z^2 = \alpha_1^2 (b_1^2 + b_2^2) + b_2^2 + b_4^2 \). These imply that
\[
z^2 = [\alpha_1^2 (b_1^2 + b_2^2) + b_2^2 + b_4^2] - [(\alpha_1 b_1 + b_4)^2 - 2\alpha_1 \alpha_2] = (1 - \alpha_1)^2 b_2^2.
\]
Hence we may assume \( \alpha_1 < 1 \). In addition, we also obtain that
\[
a + b = \alpha_1 b_1 + b_4 = \alpha_1 b_1 + 1 + \alpha_2 - b_1 = 1 + \alpha_2 - (1 - \alpha_1) b_1
\]
and hence
\[
b_1 = \frac{1}{1 - \alpha_1} (1 + \alpha_2 - a - b)
\]
\[
= \frac{1}{1 - \alpha_1} [(1 - a)(1 - b) - ab + \alpha_2]
\]
\[
= \frac{1}{1 - \alpha_1} [\alpha_2 (1 - \alpha_1) + (1 - a)(1 - b)],
\]
where the last equality follows from \( ab = \alpha_1 \alpha_2 \). Let \( c = (1 - a)(1 - b)/(1 - \alpha_1) \). Then \( b_1 = \alpha_2 + c \) and \( b_4 = 1 - c \). By a direct computation, we see that
\[
z^2 = (1 - \alpha_1)^2 b_2^2 = (1 - \alpha_1)^2 (b_1 b_4 - \alpha_2)
\]
\[
= (1 - \alpha_1)^2 [(\alpha_2 + c)(1 - c) - \alpha_2] \quad \text{(because } \alpha_2 = b_1 b_4 - b_2^2]\)
\[
= c(1 - \alpha_1)[(1 - \alpha_1)(1 - \alpha_2) - c(1 - \alpha_1)]
\]
\[
= (1 - a)(1 - b)[(a + b) - (\alpha_1 + \alpha_2)],
\]
where the last equality follows from \( c = (1 - a)(1 - b)/(1 - \alpha_1) \) and \( ab = \alpha_1 \alpha_2 \). Since \( ab = \alpha_1 \alpha_2 \), we have \( \alpha_1 + \alpha_2 \geq 2\sqrt{\alpha_1 \alpha_2} = 2\sqrt{ab} \). This implies that
\[
z \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1 - a)(1 - b)}.
\]
In general, since \( C = \alpha \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \alpha (\begin{pmatrix} A \\ \|A\| \end{pmatrix}) (\begin{pmatrix} B \\ \|B\| \end{pmatrix}) \), where \( 0 < \alpha = \|A\| \|B\| \leq 1 \), the scalars \( a, b, z \) in the above can be replaced by \( a/\alpha, b/\alpha, z/\alpha \), respectively, to get \( 0 \leq a/\alpha, b/\alpha \leq 1 \) and
\[
\frac{z}{\alpha} \leq \sqrt{(1 - a/\alpha)(1 - b/\alpha)} |\sqrt{a/\alpha} - \sqrt{b/\alpha}|.
\]
This shows that \( 0 \leq a, b \leq \alpha \leq 1 \) and
\[
z \leq |\sqrt{a} - \sqrt{b}| \sqrt{(\alpha - a)(1 - b/\alpha)} \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1 - a)(1 - b)}.
\]
This proves the necessity.

In order to prove Theorem 1.1, we need the following fact; see, for example, [9, p. 547].
Lemma 2.2. Let $A$ be a bounded linear operator of the form
\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21}^* & A_{22}
\end{pmatrix}
on H \oplus K,
\]
where $H$ and $K$ are Hilbert spaces. Then $A$ is positive if and only if $A_{11}$ and $A_{22}$ are both positive and there exists a contraction $D$ mapping $K$ into $H$ satisfying $A_{12} = A_{11}^{1/2}DA_{22}^{1/2}$.

Lemma 2.3. Suppose $a_{11}(z), a_{22}(z), a_{12}(z) = a_{21}(z)$ are continuous real-valued functions defined on $S \subseteq [0, \infty)$ such that $A = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\
a_{21}(z) & a_{22}(z) \end{pmatrix} \geq 0$ for all $z \in S$. Then $\begin{pmatrix} a_{11}(P) & a_{12}(P) \\
a_{21}(P) & a_{22}(P) \end{pmatrix} \geq 0$ on $H \oplus H$ for all positive operators $P \in B(H)$ with spectrum in $S$.

Proof. Since $A \geq 0$, we have $a_{11}(z), a_{22}(z) \geq 0$ and
\[
0 \leq a_{12}(z)a_{21}(z) \leq a_{11}(z)a_{22}(z), \quad z \in S.
\]
Define $h(z)$ by
\[
h(z) := \begin{cases} 
a_{12}(z) & \text{if } |a_{12}(z)| > 0, \\
a_{12}^{1/2}(z)a_{22}^{1/2}(z) & \text{if } a_{12}(z) = 0.
\end{cases}
\]
Then $h(z)$ is a bounded Borel function on $S$ with $|h(z)| \leq 1$, which satisfies
\[
a_{12}(z) = a_{11}^{1/2}(z)h(z)a_{22}^{1/2}(z).
\]
By the spectral theorem, for all positive operators $P \in B(H)$ with spectrum in $S$, we have $a_{11}(P) \geq 0$, $a_{22}(P) \geq 0$, $a_{12}(P) = a_{21}(P) \geq 0$ and
\[
a_{12}(P) = a_{11}^{1/2}(P)h(P)a_{22}^{1/2}(P)
\]
for the contraction $h(P) \in B(H)$. Our assertion follows from Lemma 2.2. \[\]

In the finite dimensional case, Wu [14, Corollary 2.3] has shown that if $C = \begin{pmatrix} C_1 & C_3 \\
0 & C_2 \end{pmatrix}$ is a product of two positive operators, then so are $C_1$ and $C_2$. Proposition 2.4 gives another proof which holds for both finite and infinite dimensional Hilbert spaces. In fact, it is also true that positive operators are replaced by positive contractions.

Proposition 2.4. Let $T$ be a bounded linear operator of the form
\[
\begin{pmatrix}
T_1 & T_3 \\
0 & T_2
\end{pmatrix}
on H \oplus K,
\]
where $H$ and $K$ are both Hilbert spaces. If $T$ is a product of two positive contractions, then so are $T_1$ and $T_2$.

Proof. By our assumption and Lemma 2.2, we may assume that $T = AB$, where $A$ and $B$ are of the form
\[
\begin{pmatrix}
A_1 & A_1^{1/2}D_1A_2^{1/2} \\
A_2^{1/2}D_1^*A_1^{1/2} & A_2
\end{pmatrix}\quad \text{and} \quad \begin{pmatrix}
B_1 & B_1^{1/2}D_2B_2^{1/2} \\
B_2^{1/2}D_2^*B_1^{1/2} & B_2
\end{pmatrix}\quad \text{on } H \oplus K,
\]
respectively, such that $0 \leq A_1 \leq I_H$, $0 \leq A_2 \leq I_K$, $0 \leq B_1 \leq I_H$, $0 \leq B_2 \leq I_K$, $D_1$ and $D_2$ are contractions from $K$ into $H$. From $T = AB$, we obtain that

\begin{equation}
T_1 = A_1 B_1 + A_1^{1/2} D_1 (A_2^{1/2} B_2^{1/2} D_2^{1/2} B_1^{1/2}),
\end{equation}

(2.5)

\begin{equation}
A_2^{1/2} (D_2^* A_1^{1/2} B_1^{1/2}) = -A_2^{1/2} (A_2^{1/2} B_2^{1/2} D_2^*) B_1^{1/2},
\end{equation}

(2.6)

Moreover, substitute this into (2.5) to get

\[ T_2 = (A_2^{1/2} D_2^* A_1^{1/2} B_1^{1/2}) D_2 B_2^{1/2} + A_2 B_2. \]

Let $E_1$ be the restriction of $A_2^{1/2}$ to $(\ker A_2^{1/2})^\perp$, then $E_1$ is injective. Since $0 \leq A_2^{1/2} \leq I_K$, so we can consider the (possibly unbounded) inverse $E := E_1^{-1}$: ran $A_2^{1/2} \to (\ker A_2^{1/2})^\perp$ such that $E A_2^{1/2} = P_0$, where $P_0$ is the orthogonal projection from $K$ onto ran$A_2^{1/2}$. Hence by (2.6), we derive that

\[ A_2^{1/2} B_2^{1/2} D_2 B_1^{1/2} = P_0 (A_2^{1/2} B_2^{1/2} D_2^* B_1^{1/2}) = -P_0 (D_2^* A_1^{1/2} B_1). \]

Moreover, substitute this into (2.5) to get

\[ T_1 = A_1 B_1 - A_1^{1/2} D_1 (P_0 (D_2^* A_1^{1/2} B_1)) = [A_1^{1/2} (I_H - D_1 P_0 D_2^*) A_1^{1/2}] B_1 \]

\[ = [A_1^{1/2} (I_H - (P_0 D_2^*) (P_0 D_2^*)) A_1^{1/2}] B_1. \]

Note that $\|P_0 D_1^*\| \leq 1$ implies that

\[ 0 \leq (I_H - (P_0 D_2^*) (P_0 D_1^*)) \leq I_H. \]

Therefore, $T_1 = [(A_1^{1/2} P_1^*) P_1 A_1^{1/2}] B_1$, where $P_1^* P_1 = I_H - (P_0 D_2^*) (P_0 D_1^*)$ for some positive contraction $P_1$ on $H$. This shows that $T_1$ is a product of two positive contractions. Similarly, we can show that $T_2^*$ is a product of two positive contractions, and hence so is $T_2$. This completes our proof.

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We first prove the necessity. By assumption, we can focus on the part

\[ \begin{pmatrix}
  aI \\
  0
\end{pmatrix} \begin{pmatrix}
  P \\
  bI
\end{pmatrix} \in B(H_3 \oplus H_3) \]

for some $P > 0$. Now, consider a $2 \times 2$ matrix

\[ \begin{pmatrix}
  a \\
  0 \\
  z
\end{pmatrix} \]

with $a, b \in [0, 1]$ and

\[ z \in S := \{c : 0 \leq c \leq |\sqrt{a} - \sqrt{b}| \sqrt{(1-a)(1-b)}\}. \]

Then by Lemma 2.1, there are continuous maps $a_{ij}(z), b_{ij}(z)$ for $1 \leq i, j \leq 2$ with $a_{12}(z) = a_{21}(z) \geq 0$, $b_{12}(z) = b_{21}(z) \leq 0$ and satisfy

\[ 0 \leq (a_{ij}(z)) \leq I_2, \quad 0 \leq (b_{ij}(z)) \leq I_2, \quad (a_{ij}(z)) (b_{ij}(z)) = \begin{pmatrix}
  a \\
  0 \\
  z
\end{pmatrix}, \quad z \in S. \]
By Lemma 2.3,
\[ 0 \leq (a_{ij}(P)) \leq I \quad \text{and} \quad 0 \leq (b_{ij}(P)) \leq I. \]

By the spectral theorem on positive operators,
\[ (a_{ij}(P))(b_{ij}(P)) = \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix}. \]

To prove the converse, suppose there is a factorization of the quadratic operator \( T \in B(H) \) with operator matrix \( aI \oplus bI \oplus \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \) for some \( P \geq 0 \) as the product of two positive contractions.

By Proposition 2.4, we know that
\[ T_1 = \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} = AB \quad \text{for some} \quad 0 \leq A, B \leq I, \ A, B \in B(H_3 \oplus H_3). \]

We may use the Berberian construction (see [6]) to embed \( H_3 \) into a larger Hilbert space \( K_3, B(H_3) \) into \( B(K_3) \). Suppose \( A = (A_{ij})_{1 \leq i,j \leq 2}, B = (B_{ij})_{1 \leq i,j \leq 2} \in B(H_3 \oplus H_3) \). Then \( P, A, \) and \( B \) are extended to \( \tilde{P} \in B(K_3), \tilde{A} = (\tilde{A}_{ij})_{1 \leq i,j \leq 2} \in B(K_3 \oplus K_3), \) and \( \tilde{B} = (\tilde{B}_{ij})_{1 \leq i,j \leq 2} \in B(K_3 \oplus K_3), \) respectively, such that the following conditions hold.

(a) \( \tilde{P} \geq 0 \) with \( \|P\| = \|\tilde{P}\| \) such that all the elements in \( \sigma(\tilde{P}) \) are eigenvalues of \( \tilde{P} \).

(b) \( 0 \leq \tilde{A}, \tilde{B} \leq I \) such that \( \tilde{T}_1 = \begin{pmatrix} aI & \tilde{P} \\ 0 & bI \end{pmatrix} = \tilde{A}\tilde{B} \).

Since \( \tilde{P} \geq 0 \) and \( \sigma(\tilde{P}) \) are eigenvalues of \( \tilde{P} \), the quadratic operator \( \tilde{T}_1 \) is unitarily similar to \( \begin{pmatrix} a & \|P\| \\ 0 & b \end{pmatrix} \oplus T_2 \) that admits a factorization as the product of two positive contractions. By Proposition 2.4, we see that \( \begin{pmatrix} a & \|P\| \\ 0 & b \end{pmatrix} \) is a product of two positive contractions. Thus,
\[ \|P\| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1 - a)(1 - b)}. \]

**Remark 2.5.** Inspired by a comment of the referee, we see that if one considers the set of operators of the form \( \begin{pmatrix} aI & P \\ 0 & bI \end{pmatrix} \) with respect to a fixed orthonormal basis, then our proof of Theorem 1.1 shows that the decomposition depends continuously on \( P \), and therefore continuous on \( T \).

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