Math 211  Sample Final Examination questions  Name:__________________  

Answer 10 out of 11 questions.  

1. (a) Determine $h$ so that $b = \begin{bmatrix} 1 \\ 4 \\ h \end{bmatrix}$ lies in the linear span of the set \( \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \).  

(b) Show that every $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ lies in the span of the set \( \left\{ \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \).  

2. Suppose $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$, and $\det(A - \lambda I) = (1 + \lambda)^2(5 - \lambda)$. Determine an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix.  

3. Find the orthogonal projection of $y = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ on the subspace $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.  

4. Let $T : \mathbb{P}_2(t) \to \mathbb{R}^2$ be the transformation $T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.  

(a) Determine the matrix of $T$ relative to the standard bases $C_1 = \{1, t, t^2\}$, $C_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.  

(b) Determine the matrix of $T$ relative to the standard bases $\tilde{C}_1 = \{1, t, t-t^2\}$, $\tilde{C}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.  

5. Let $T : \mathbb{P}_1(t) \to \mathbb{P}_1(t)$ defined by $T(a_0 + a_1t) = (a_0 - 2a_1) + (a_0 + 4a_1)t$.  

(a) Find the matrix for $T$ relative to the standard basis $C = \{1, t\}$.  

(b) For $i = 1, 2, \ldots$, find eigenvalue $\lambda_i$ and eigenvector $u_i \in \mathbb{P}_1(t)$ such that $T(u_i) = \lambda_i u_i$.  

6. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$.  

(a) Apply the Gram-Schmidt process to the columns of $A$ to get an orthonormal basis for $\mathbb{R}^3$.  

(b) Find orthogonal matrix $Q$ and upper triangular matrix $R$ such that $A = QR$.  

7. Let $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$. Find the least-squares solution of $Ax = b$, and compute the associated least-squares error.  

8. Suppose $A$ is $m \times n$ and $B$ is $n \times p$.  

(a) Show that $\text{Col}(AB)$ is a subspace of $\text{Col}A$.  

[It suffices to show that $\text{Col}(AB)$ is a subset of $\text{Col}A$.]  

(b) If $A$ and $AB$ has the same rank, show that $A$ and $AB$ have the same column space.  

9. Suppose $W$ is a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{u_1, \ldots, u_p\}$ and $W^\perp$ has an orthogonal basis $\{v_1, \ldots, v_q\}$.  

(a) Show that $B = \{u_1, \ldots, u_p, v_1, \ldots, v_q\}$ is an orthogonal set.  

(b) Show that $B$ is a basis for $\mathbb{R}^n$, and deduce that $p + q = n$.  

---

1
10. Suppose $A$ is a $4 \times 4$ with eigenvalues $1, -1, 0$, and $P$ is invertible such that $P^{-1}AP = D$ where $D$ is a diagonal matrix with diagonal entries $d_1 \geq d_2 \geq d_3 \geq d_4$.
   (a) Determine all possible form of $(d_1, d_2, d_3, d_4)$.
   (b) Determine $A^{2014}$ and $A^{2015}$ in terms of $A$.

11. Construct example of $2 \times 2$ matrices satisfying the following conditions.
   (a) $A$ has no real eigenvalues.
   (b) $B$ has real eigenvalue(s) but not diagonalizable.
   (c) $C$ and $P$ such that $P$ is invertible and $P^{-1}CP \neq C$. 