Objective:

Study basic matrix operations and invertible matrices.
An $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns, often denoted by $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$.

- One can discuss the $(i, j)$, the diagonal entries, etc.
- We often use diagonal matrices, identity matrix, zero matrix, etc.
Theorem 1 Let $A, B, C$ be matrices of the same size, $r, s$ be scalars. Then

(a) $A + B = B + A$, 
(b) $(A + B) + C = A + (B + C)$,
(c) $A + O = O + A$, 
(d) $r(A + B) = rA + rB$,
(e) $(r + s)A = rA + sA$, 
(f) $(rs)A = r(sA)$.

Proof. Note that the $(i, j)$ entry of $X + Y = x_{ij} + y_{ij}$, and $X = Y$ if they have the same $(i, j)$ entries for each $(i, j)$ pair.

Therefore, ....
Recall: If $A$ has columns $v_1, \ldots, v_n$ and $x \in \mathbb{R}^n$ has entries $x_1, \ldots, x_n$, then $Ax = x_1v_1 + \cdots + x_nv_n$.

**Definition (Matrix multiplication)**

Let $A$ be an $m \times n$ matrix, and $B = [b_1 | \cdots | b_p]$ be an $n \times p$ matrix. Then

$$AB = [Ab_1 | \cdots | Ab_p]$$

is an $m \times p$ matrix.

**Examples** ...

**Remark** If $A = \begin{bmatrix} \text{row}_1(A) \\ \vdots \\ \text{row}_n(A) \end{bmatrix}$ then $AB = \begin{bmatrix} \text{row}_1(AB) \\ \vdots \\ \text{row}_n(AB) \end{bmatrix}$.

**Proof.** Check the $(i, j)$ entry!
Theorem 1

Let $A, B, C$ be matrices of suitable sizes, $r$ be a scalar. Then

(a) $(AB)C = A(BC)$  
(b) $A(B + C) = AB + AC$,

(c) $(B + C)A = BA + CA$,  
(d) $r(AB) = (rA)B$,

(e) $I_mA = A = AI_n$  if $A$ is $m \times n$.

Proof. (a) Let $A = [A_1 \cdots A_n]$ be $m \times n$, $B = [B_1 \cdots B_p]$ be $n \times p$, and $C = [C_1 \cdots C_q] = [c_{ij}]$ be $p \times q$. Then $AB = [Ab_1|\cdots|Ab_p]$.

We show that the $k$th column of $(AB)C$ equals to that of $A(BC)$...

(b) We show that the $k$th column of $A(B + C)$ equals that of $AB + AC$...

Remarks and Examples

(1) $AB \neq BA$ in general.

(2) $AB = AC$ does not imply that $B = C$.

(3) $AB = 0$ does not imply $A = 0$ or $B = 0$.  

Linear Algebra: Section 2.1 - 2.2
More terminology/operations

If \( v \in \mathbb{R}^n \) has entries \( v_1, \ldots, v_n \), then the transpose of \( v \) is \( v^T = [v_1 \cdots v_n] \).

Let \( A = [A_1 \cdots A_n] \) be \( m \times n \). Then the transpose of \( A \) is \( A^T \) is \( n \times m \) with rows \( A_1^T, \ldots, A_n^T \).

If \( A \) is \( n \times n \), then \( A^r = A \cdots A \) for any positive integer \( r \).

**Theorem 3** Let \( A, B \) be matrices of appropriate sizes, \( r \) be a scalar.

(a) \( (A^T)^T = A \).
(b) \( (A + B)^T = A^T + B^T \),
(c) \( (rA)^T = rA^T \),
(d) \( (AB)^T = B^T A^T \).

Proof of (d). The \((i, j)\) entry of \( B^T A^T \) equals \( \text{Row}_i(B)\text{Col}_j(A) \).

The \((i, j)\) entry of \( (AB)^T \) is the \((j, i)\) entry of \( AB \) and equals

\( \text{Row}_j(A)\text{Col}_i(B) \).
**Definition** A square matrix $A$ is invertible if there is a matrix $B$ such that

$$AB = BA = I.$$ 

If such a $B$ exists, it is unique, and is called the inverse of $A$, denoted by $A^{-1}$.

**Theorem 4** Let $A = [a_{ij}]$ be $2 \times 2$. If $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}.$$ 

**Proof.** Direct checking.

**Theorem 5** If $A$ is an invertible $n \times n$ matrix, then for each $b \in \mathbb{R}^n$ the linear system $Ax = b$ has a unique solution $x = A^{-1}b$.

**Proof.** If $Ax = b$, then $x = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}b$.

If $x = A^{-1}b$, then $Ax = A(A^{-1}b) = (AA^{-1})b = b$. 

Theorem 7  Row reductions on \([A|I_n]\) yield \([I_n|A^{-1}]\) if \(A\) is invertible.

Proof. Let \(A^{-1} = B = [B_1 \cdots B_n]\).

Then \(AB_k = e_k\) for \(k = 1, \ldots, n\),
and \(B_k\) is obtained by row reduction of \([A|e_k]\) to \([I_n|B_k]\).
Thus, we may row reduce \([A|I_n]\) to \([I_n|B_1 \cdots B_n] = [I_n|B] = [I_n|A^{-1}]\).

Example ....
Theorem 6 Let $A, B$ be invertible $n \times n$ matrices.

(a) $(A^{-1})^{-1} = A$.
(b) $(AB)^{-1} = B^{-1}A^{-1}$ (mind the order).
(c) $(A^T)^{-1} = (A^{-1})^T$.

Proof. (a) Let $C = (A^{-1})^{-1}$ so that $I = A^{-1}C$.

Multiplying both sides by $A$, we get $A = C$.

(b) Let $C = (AB)^{-1}$ so that $I = (AB)C$.

Multiplying both sides by $B^{-1}A^{-1}$, we see that $B^{-1}A^{-1} = C$.

(c) Let $C = (A^T)^{-1}$ so that $I = A^TC$. Multiplying both sides by $(A^{-1})^T$,
we see that $(A^{-1})^T = (A^{-1})^TA^TC = (AA^{-1})^TC = C$. 
There are three types of elementary matrices obtained from $I_n$.

1) Interchanging the $i$th and $j$th row (of the identity matrix).
2) Adding $r \neq 0$ times the $i$th row to the $j$th row (of the identity matrix).
3) Multiplying the $i$th row by $r \neq 0$.

**Remark**

If $E$ is an elementary matrix, then $EA$ corresponds to the elementary row operation to $A$. We can use this to prove that $AB = I$ ensures $BA = I$.

**Theorem** If there are elementary operations $E_1, \ldots, E_k$ such that

$$E_k \cdots E_1 [A| I] = [I_n | E_k \cdots E_1] = [I | B],$$

then $BA = I = AB$.

**Proof.** Applying the row reduction to $A$ to get $I_n$ is the same as multiplying $A$ with the corresponding elementary matrices. So, $BA = E_k \cdots E_1 A = I_n$. Note that $AB_i = e_i$ for each $i = 1, \ldots, n$. Thus $AB = I_n$.\hfill \square