1. (25 points, 5 points each) Give brief and complete answers to the following:

(a) Let \( A = \{ a, b, c \} \) and \( B = \{ \varnothing, \{ \varnothing \} \} \). Find \( A \times B \) and \( \mathcal{P}(B) \) (power set of \( B \)).
Solution. \( A \times B = \{(a, \varnothing), (b, \varnothing), (c, \varnothing)\} \) and \( \mathcal{P}(B) = \{ \varnothing, \{ \varnothing \}, \{ \{ \varnothing \} \}, B \} \).

(b) State the contrapositive of the following statement: “If \( n \) is an odd integer, then \( n^3 \) is divisible by 3.”
Solution. If \( n \) is an integer such that \( n^3 \) is not divisible by 3, then \( n \) is even.

(c) State the negation of the following statement:
“For any \( a \geq 0 \), there exists \( b \) with \( 0 \leq b \leq 1 \), so that for any \( c < 0 \), \( ab = c \).”
Solution. There is \( a \geq 0 \) such that for every \( b \in [0, 1] \) there exists \( c < 0 \) satisfying \( ab \neq c \).

(d) Prove or disprove the following statement: There exists an integer \( x \) such that \( x^2 \equiv 3 \pmod{4} \).
Solution. The statement is false. Consider two cases. If \( x = 2k \) even, then \( x^2 \equiv 4k^2 \equiv 0 \pmod{4} \); if \( x = 2k + 1 \) is odd, then \( x^2 \equiv 4k^2 + 4k + 1 \equiv 1 \pmod{4} \). Thus, there is no integer \( x \) such that \( x^2 \equiv 3 \pmod{4} \).

(e) Let \( P \) and \( Q \) be two statements. Construct a truth table for \( P \land (Q \Rightarrow \lnot P) \).
Solution.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \lnot P )</th>
<th>( (Q \Rightarrow \lnot P) )</th>
<th>( P \land (Q \Rightarrow \lnot P) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

A different solution. If \( P \) is false, then the statement is false. If \( P \) is true and \( Q \) is true, the statement is false. So, the only case for the statement to be true is when \( P \) is true, and \( Q \) is false.

2. (15 points) Let \( A, B \) be sets. Show that \( A \cup B = B \) if and only if \( A \subseteq B \).
Solution. Assume \( A \cup B = B \). Then \( A \subseteq A \cup B = B \). Assume that \( A \subseteq B \). Then \( A \cup B \subseteq B \) and \( B \subseteq A \cup B \); hence \( A \cup B = B \).

3. (15 points) Show that \( 3\sqrt{2} = 2^{1/3} \) is irrational. [You cannot use the Fundamental Theorem of Arithmetic.]
Solution. Proof by contradiction. Assume that \( 2^{1/3} = m/n \) for some \( m, n \in \mathbb{N} \) so that \( \text{gcd}(m, n) = 1 \). Then \( 2n^3 = m^3 \). Then \( m \) cannot be odd. Else, \( m^3 \) is odd. So, \( m = 2k \) for some \( k \in \mathbb{N} \). It follows that \( 2n^3 = 8k^3 \) so that \( n^3 = 4k^3 \). Again, \( n \) cannot be odd. Else, \( n^3 \) is odd. So, \( n \) is also even as \( m \) is, which contradicts the fact that \( \text{gcd}(m, n) = 1 \).

4. (15 points) Prove that \( 8|\left(7^{2n} - 1\right) \) for every nonnegative integer \( n \).
Solution. When \( n = 0 \), we have \( 8|0 \). The statement is true.
Assume the statement holds for \( n = k \), i.e., \( 7^{2k} - 1 = 8m \) for some \( m \in \mathbb{Z} \). Then for \( n = k + 1 \),
\[
7^{2(k+1)} - 1 = 49 \cdot 7^{2k} - 1 = 49(8m + 1) - 1 = 8(49m + 6),
\]
which is a multiple of 8. Thus, the statement also holds for \( n = k + 1 \).
By the principle of MI, the statement holds for all nonnegative integer \( n \).

5. (15 points) Show that for any \( n \in \mathbb{N} \), \( n^2 \) cannot be of form \( 5m + 2 \) or \( 5m + 3 \), where \( m \) is an integer.
Solution. Suppose \( n = 5m + r \) with \( r = 0, 1, 2, 3, 4 \). Then \( n^2 = 25m^2 + 10mr + r^2 \equiv s \pmod{5} \) for \( s = 0, 1, 4, 5 \) depending on \( r = 0, 1, 2, 3, 4 \). The result follows.

6. (15 points) Suppose \( S_\alpha = (-1 - \alpha, 1 + \alpha) \) for \( \alpha > 0 \). Prove that \( \cap_{\alpha \in (0,1)} S_\alpha = [-1, 1] \).
Solution. Note that \( [-1, 1] \subseteq (-1 - \alpha, 1 + \alpha) \) for all \( \alpha \in (0, 1) \). Thus, \( [-1, 1] \subseteq \cap_{\alpha \in (0,1)} S_\alpha \). To prove the reverse inclusion, suppose \( x \in \cap_{\alpha \in (0,1)} S_\alpha \). We show that \( x \in [-1, 1] \). If it is not true, then \( |x| > 1 \). Let \( \beta = \min\{|x| - 1\}/2, 1/2\} \in (0, 1) \). Then \( |x| = 2\beta + 1 > \beta + 1 \) so that \( x \notin S_\beta \). So, \( x \notin \cap_{\alpha \in (0,1)} S_\alpha \).