1. To prove \( \cap_{\alpha \in (0,1)} S_\alpha = \{0\} \).

   Note that \( 0 \in (-\alpha, \alpha) = S_\alpha \) for every \( \alpha \in (0,1) \). Thus, \( \{0\} \subseteq \cap_{\alpha \in (0,1)} S_\alpha \).

   For the reverse inclusion, we need to show that if \( x \in \cap_{\alpha \in (0,1)} S_\alpha \) then \( x = 0 \). We show the contrapositive and consider two cases.

   Case 1. If \( x \) is such that \( |x| \geq 0.5 \), then \( x \notin (-0.5, 0.5) = S_{0.5} \) so that \( x \notin \cap_{\alpha \in (0,1)} S_\alpha \).

   Case 2. If \( 0 < |x| < 0.5 \), then \( x \notin (-x, x) = S_x \) so that \( x \notin \cap_{\alpha \in (0,1)} S_\alpha \).

   Next, we prove \( \cup_{\alpha \in (0,1)} S_\alpha = (-1,1) \). If \( x \in \cup_{\alpha \in (0,1)} S_\alpha \), then \( x \in (-\alpha, \alpha) = S_\alpha \) for some \( \alpha \in (0,1) \). Thus, \( x \in (-1,1) \).

   To prove the reverse inclusion, suppose \( x \in (-1,1) \). Let \( \beta = (1 + |x|)/2 \). Then \( |x| \leq \beta \) so that \( x \in S_\beta \). Hence, \( x \in \cup_{\alpha \in (0,1)} S_\alpha \).

2. If \( U \) is the universal set, then
   \[ (A - B) \cup (A \cap B) = (A \cap \overline{B}) \cup (A \cap B) \quad \text{by the definition of } A - B \]
   \[ = A \cap (\overline{B} \cup B) \quad \text{by the distributive law} \]
   \[ = A \cap U \quad \text{U is the universal set} \]
   \[ = A. \]

3. For any three sets \( A, B, C \), show that \( (A \times C) - (B \times C) \subseteq (A - B) \times C \).

   Solution. Suppose \((x, y) \in (A \times C) - (B \times C)\). Then \((x, y) \in A \times C \) and \((x, y) \notin B \times C \).

   So, \((x \in A, y \in C) \) and it is not true that \((x \in B, y \in C) \). Since we know that \( y \in C \), so \( x \notin B \).

   Thus, we have \( x \in A, x \notin B, y \in C \). Hence, \( x \in (A \cap \overline{B}) = (A - B) \) and \( y \in C \), i.e., \((x, y) \in (A - B) \times C \).

   Suppose \((x, y) \in (A - B) \times C \). Then \( x \in A, x \notin B, y \in C \). So, \((x, y) \in A \times C \) and \((x, y) \notin B \times C \), i.e., \((x, y) \in (A \times C) - (B \times C) \).

4. Prove that if \( a, b, c, d \in \mathbb{R} \), then \((ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)\).

   Solution. Note that
   \[ (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 = (ab)^2 + (ad)^2 + (cb)^2 + (cd)^2 - (a^2b^2 + 2abcd + c^2d^2) \]
   \[ = (ad)^2 - 2abcd + (bc)^2 = (ad - bc)^2 \geq 0. \]

   So, \((a^2 + c^2)(b^2 + d^2) \geq (ab + cd)^2\).

5. Prove that if \( a, b, c \in \mathbb{R} \), then \(|x - z| \leq |x - y| + |y - z|\).

   Solution. Let \( a = x - y \) and \( b = y - z \). Then \( a + b = (x - y) + (y - z) = (x - z) \).

   We need to prove \(|a + b| \leq |a| + |b| \). Note that \(|p| \leq L \) for a positive number \( L \) if and only if \(-L \leq p \leq L \). Now, \(-|a| \leq a \leq |a| \) and \(-|b| \leq b \leq |b| \). So, \(-(|a| + |b|) \leq a + b \leq |a| + |b| \).

   Thus, \(|a + b| \leq |a| + |b| \).
6. Let $n \in \mathbb{Z}$. Prove that $3|\left(2n^2 + 1\right)$ if and only if $3 \not| n$.

Solution. If $3 \not| n$, then $n = 3k + 1$ or $3k + 2$. Thus,

$$2n^2 + 1 = 2(3k + 1)^2 + 1 = 2(9k^2 + 6k + 1) + 1 = 3(6k^2 + 4k + 1)$$

or

$$2n^2 + 1 = 2(3k + 2)^2 + 1 = 2(9k^2 + 6k + 4) + 1 = 3(6k^2 + 4k + 3).$$

In either case, $2n^2 + 1$ is divisible by $3$.

To prove the converse (reverse implication), we use the indirect proof, i.e., prove the contrapositive of the statement. Assume that $3|n$ so that $n = 3k$. Then

$$2n^2 + 1 = 18k^2 + 1 = 3(6k^2) + 1$$

is not divisible by $3$.

7. Let $a, b \in \mathbb{Z}$ satisfy $a^2 + 2b^2 \equiv 0 \pmod{3}$. Prove that either $a \equiv b \equiv 0 \pmod{3}$ or neither $a$ nor $b$ is congruent to $0$ modulo $3$.

Solution. We use indirect proof, i.e., prove the contrapositive. Suppose it is not true that

(i) $a \equiv b \equiv 0 \pmod{3}$, or (ii) $a \not\equiv 0 \pmod{3}$ and $b \not\equiv 0 \pmod{3}$.

That is

“one of the numbers $a$ and $b$ is divisible by $3$, and the other is not divisible by $3.”$.

We may assume that without loss of generality that $a = 3k$ and $b = 3q \pm 1$ with $k, q \in \mathbb{Z}$.

Then $a^2 + b^2 = 9k^2 + 9q^2 \pm 6q + 1 = 3(3k^2 + 3q^2 \pm 2q) + 1$, which is not divisible by $3$.

8. Prove that if $n \in \mathbb{Z}$ is such that $n \equiv 3 \pmod{7}$, then $n^2 \equiv 2 \pmod{7}$.

Solution. Suppose $n \equiv 3 \pmod{7}$, i.e., $n = 7k + 3$. Then

$$n^2 = (7k + 3)^2 = 49k^2 + 42k + 9 = 7(7k^2 + 6k + 1) + 2$$

so that $n^2 \equiv 2 \pmod{7}$.