1. For each of the following sets, determine whether it is well-ordered and show your reasons.

(a) \( S = \{ n \in \mathbb{N} : n \text{ is even} \} \). (b) \( T = \{ x \in \mathbb{Q} : x \geq 0 \} \).

**Solution.** (a)\( S \) is well-ordered. Any nonempty subset of \( S \) is a subset of \( \mathbb{N} \) and hence has a minimum element because \( \mathbb{N} \) is well-ordered.

(b) \( T \) is not well-ordered because \( T_1 = \{ x \in \mathbb{Q} : x > 0 \} \) is a subset of \( T \) that has no smallest element. To see this, note that \( 1 \in T_1 \) so that \( T_1 \) is nonempty; for any \( x \in T_1 \) we have \( y = x/2 \in T_1 \) so that \( x \) cannot be a least element.

2. Prove that \( \sum_{k=0}^{n}(2k+1) = (n+1)^2 \) for all \( n \in \mathbb{N} \).

**Proof.** LHS of \( P(1) \) is \( 1 + (2 + 1) = 4 = (1 + 1)^2 \), which is the RHS of \( P(1) \). So, \( P(1) \) holds.

Assume that \( P(m) \) holds for \( m \geq 1 \), i.e., \( \sum_{k=0}^{m}(2k+1) = (m+1)^2 \).

Consider \( P(m+1) \). We have

\[
\sum_{k=0}^{m+1}(2k+1) = \sum_{k=0}^{m}(2k+1) + [2(m+1)+1]
\]

\[
= (m+1)^2 + 2(m+1) + 1 = ((m+1)+1)^2 = (m+2)^2.
\]

So, \( P(m+1) \) holds. By the principle of mathematical induction, \( P(n) \) holds for all \( n \in \mathbb{N} \).

3. Prove that \( \sum_{k=1}^{n}\frac{1}{(k+2)(k+3)} = \frac{n}{3n+9} \) for every positive integer \( n \).

**Proof.** When \( n = 1 \), we have \( \frac{1}{(1+2)(1+3)} = \frac{1}{12} = \frac{1}{3+3} \). So, \( P(1) \) holds.

Suppose \( P(m) \) holds, i.e., \( \sum_{k=1}^{m}\frac{1}{(k+2)(k+3)} = \frac{m}{3m+9} \). Consider \( P(m+1) \). We have

\[
\sum_{k=1}^{m+1}\frac{1}{(k+2)(k+3)} = \frac{m}{3m+9} + \frac{1}{(m+3)(m+4)}
\]

by induction assumption

\[
= \frac{1}{m+3} \left( \frac{m}{3} + \frac{1}{m+4} \right) = \frac{1}{m+3} \frac{m(m+4)+3}{3(m+4)}
\]

\[
= \frac{1}{m+3} \frac{m^2 + 4m + 3}{3(m+4)} = \frac{1}{m+3} \frac{(m+3)(m+1)}{3(m+4)} = \frac{m+1}{3(m+1)+9}
\]

So, \( P(m+1) \) is true. By the principle of mathematical induction, \( P(n) \) holds for all \( n \in \mathbb{N} \).

4. Determine (with proof) the set of integers \( n \) such that \( n \geq 3, n^3 \leq 3^n \).

**Proof.** When \( n = 3 \), \( n^3 = 3^3 = 3^n \). So, \( P(3) \) holds.

Assume that \( P(k) \) holds for \( k \geq 3 \), \( k^3 \leq 3^k \) for some \( k \geq 3 \). Consider \( P(k+1) \). We have

\[
3^{k+1} = 3 \cdot 3^k \geq 3 \cdot (k^3)
\]

by induction assumption

\[
= k^3 + k^3 + k^3 \geq k^3 + 3k^2 + 3^2k \quad \text{because} \quad k \geq 3
\]

\[
\geq k^3 + 3k^2 + 3k + 1 = (k+1)^3 \quad \text{because} \quad 9k \geq 3k + 1
\]

So, \( P(k+1) \) holds. By the principle of mathematical induction, \( n^3 \leq 3^n \) for all \( n \geq 3 \).
5. Prove \( P(n) : 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n} \) for all \( n \in \mathbb{N} \).

**Proof.** When \( n = 1 \), we have \( 1 \leq 1 - 1/1 \). So, \( P(1) \) holds.

Suppose \( P(k) \) holds, i.e., \( 1 + \frac{1}{4} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k} \). Consider \( P(k+1) \). By induction assumption,

\[
1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}
\]

because

\[
2 - \frac{1}{k+1} - \left( 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \right) = \frac{(k+1)^2 - k(k+1) - k}{k(k+1)^2} = \frac{1}{k(k+1)^2} > 0.
\]

Thus, \( P(k+1) \) holds. By the principle of mathematical induction, the result follows.

6. Prove \( 7|(3^{2n} - 2^n) \) for all \( n \in \mathbb{N} \).

**Proof.** We prove the result for all \( n \in \mathbb{N} \cup \{0\} \). When \( n = 0 \), we see that \( 3^0 - 2^0 = 0 \) is divisible by 7. So, \( P(0) \) holds.

Suppose \( P(m) \) holds for \( m \geq 0 \). That is, \( 3^2m - 2^m = 7r \) for some \( r \in \mathbb{Z} \). Consider \( P(m+1) \).

Then

\[
3^{2(m+1)} - 2^{m+1} = 3^{2m} \cdot 3^2 - 2^m \cdot 2 = 3^m \cdot 7 + 2(3^m - 2^m) = 7(3^m + 2r)
\]

is a multiple of 7. Thus, \( P(m+1) \) holds.

By the principle of mathematical induction, \( P(n) \) holds for all \( n \in \mathbb{N} \cup \{0\} \).

7. Prove that \( 12|(n^4 - n^2) \) for all \( n \in \mathbb{N} \cup \{0\} \).

**Solution.** If \( n = 0 \), we have 12|0 so that the statement is true.

Suppose \( 12|(m^4 - m^2) \) for an integer \( m \geq 0 \). Then \( m^4 - m^2 = 12q \) for some \( q \in \mathbb{Z} \). Note that

\[
(m+1)^4 - (m+1)^2 = m^4 + 4m^3 + 6m^2 + 4m + 1 - m^2 - 2m - 1 = m^4 - m^2 + (4m^3 + 6m^2 + 2m) = 12q + 2m(m+1)(m+2).
\]

Consider three cases,

(a) \( m = 3k \): \( m(m+1)(m+2) = 3k(m+1)(m+2) = 6kL_1 \) as \( (m+1)(m+2) = 2L_1 \) is even;

(b) \( m = 3k + 1 \): \( m(m+1)(m+2) = m(m+1)3(k+1) = 3(k+1)m(m+1) = 6L_2 \)

as \( (m+1)(m+2) = 2L_2 \) is even;

(c) \( m = 3k + 2 \): \( m(m+1)(m+2) = m3(k+1)(m+2) = 6L_3m(m+2) \) if \( k+1 = 2L_3 \) is even,

or \( m(m+1)(m+2) = 6L_4(k+1)(m+2) \) with \( m = 2L_4 \) is even if \( k \) is even.

So, in all cases, \( m(m+1)(m+2) \) is divisible by 6. Thus, \( 2m(m+1)(m+2) \) is divisible by 12, and \( (m+1)^4 - (m+1)^2 \) is divisible by 12. By the principle of mathematical induction, \( 12|(n^4 - n^2) \) for all nonnegative integer \( n \).

**Remark** It is relatively easy to show that \( 3|(m^4 - m^2) \) and \( 4|(m^4 - m^2) \). One may then conclude that \( 12|(m^4 - m^2) \). However, to get this conclusion, we need a result saying that if \( a \) and \( b \) have no common factor such that \( a|n \) and \( b|n \), then \( ab|n \). This result has not been proved rigorously so that our proof avoid that.
8. A sequence \( \{a_n\} \) is defined recursively by \( a_1 = 1, a_2 = 4, a_3 = 9, \) and 
\[
a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)
\]
for \( n \geq 4 \). Conjecture a formula for \( a_n \) and prove that your conjecture is correct.

**Proof.** Examining the first 4 or 5 terms, we conjecture that 
\[ P(n) : a_n = n^2 \]
for \( n \in \mathbb{N} \).
Clearly, \( P(1), P(2), P(3) \) hold.

Suppose \( P(k) \) holds for \( k = 1, \ldots, m \) for \( m \geq 3 \). **Note that we need to ensure the first 3 cases hold to prove the next case.** Consider \( P(m+1) \). We have
\[
a_{m+1} = a_m - a_{m-1} + a_{m-2} + 2(2(m+1) - 3) \\
= m^2 - (m-1)^2 + (m-2)^2 + 2(2m - 1) \quad \text{by induction assumption} \\
= m^2 - (m^2 - 2m + 1) + (m^2 - 4m + 4) + 2(2m - 1) \\
= m^2 + 2m + 1 = (m + 1)^2.
\]
Thus, \( P(m+1) \) holds. By the general principle of mathematical induction, \( P(n) \) holds for all \( n \in \mathbb{N} \).

9. Show that an positive integer is a multiple of 9 if and only if the sum of all digits of the integer is a multiple of 9.

**Proof.** Suppose \( x = a_m \cdot \cdots a_1 a_0 = a_m 10^m + \cdots + a_1 10^1 + a_0 \), where \( a_0, \ldots, a_m \in \{0, 1, \ldots, 9\} \) such that \( a_m \neq 0 \). Note that for \( k \in \mathbb{N} \), we have 
\[
10^k \equiv 1 + 9 \cdot \underbrace{1 \cdot \cdots 1}_n \pmod{9}.
\]
Thus, \( x = a_m 10^m + \cdots + a_1 10^1 + a_0 \equiv a_m + \cdots + a_0 \pmod{9} \).

**Remark** Here we may or may use induction to prove \( 10^n \equiv 1 \pmod{9} \).