Let $A, B$ be non-empty sets. If there is an injection $f : A \to B$, then we write $|A| \leq |B|$.

If there is an injection from $A$ to $B$, but there is no bijection from $A$ to $B$, we write $|A| < |B|$.

1. (2 points for each part.) Determine with proofs the following sets are finite, denumerable or uncountable:
   (a) $\{ (x, y) : x, y \in \mathbb{N}, \text{ and } x + y = 1000 \}$.
   (b) All the functions from $\mathbb{Z}_3$ to $\mathbb{R}$.
   (c) The set of all functions from $\mathbb{N}$ to $\{0, 1\}$.

2. (2 points for each part.) Let $A_1, A_2, B_1, B_2$ be non-empty sets such that $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Suppose $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ are functions. Define $f : A_1 \cup A_2 \to B_1 \cup B_2$ by
   
   $$f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1, \\ f_2(x) & \text{if } x \in A_2. \end{cases}$$

   (a) Show that $f$ is a well-defined function.
   (b) If $f_1, f_2$ are injective, show that $f$ is injective.
   (c) If $f_1, f_2$ are surjective, show that $f$ is surjective.

3. (4 points) Let $A = \mathbb{N} - \{ n^2 : n \in \mathbb{N} \}$. Construct a bijection from $A$ to $\mathbb{N}$. Verify your answer.

4. Let $A = \{ a_1, a_2, \ldots \}$ be a denumerable set.
   (a) (3 point) Prove that for every $n \in \mathbb{N}$, $A$ can be partitioned into $n$ denumerable sets.
   (b) (3 point) Prove that $A$ can be partitioned into infinitely many denumerable sets.

5. (2 points for each part.) Suppose $A \subseteq B$. Prove or disprove the following.
   (a) If $B$ is denumerable, then $A$ is denumerable.
   (b) If $A$ is denumerable, then $B$ is denumerable.
   (c) If $B$ is uncountable, then $A$ is uncountable.
   (d) If $A$ is uncountable, then $B$ is uncountable.

6. (4 points) Prove that the set of infinite subsets of $\mathbb{N}$ is uncountable.
   [Hint: You may use the result in Homework 10, Problem 4.]

7. (4 points) Let $\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$ ($n$ times). Show that $|\mathbb{N}^n| = |\mathbb{N}|$ for any $n \in \mathbb{N}$.

8. (Extra 4 points) Construct an example of a set $A$ with subsets $B$ and $C$ such that

   $$|\mathbb{N}| < |C| < |B| < |A|.$$