Prove: \( \text{Let } A, B \text{ be sets. Then } A \cup B = A \cap B. \text{ If and only if } A = B. \)

**Proof.** \((\Rightarrow)\) Assume \( A \cup B = A \cap B. \)

Then \( A \subseteq A \cup B = A \cap B \subseteq B \)

Also \( B \subseteq A \cup B = A \cap B \subseteq A. \)

\[ \therefore A = B \]

\((\Leftarrow)\) Assume \( A = B. \)

Then \( A \cup B = A \cup A = A \)

\( A \cap B = A \cap A = A \)

\[ \therefore A \cup B = A \cap B \]
Example  \( A, B \) are sets. \( C \neq \emptyset \).

Prove \( A \times C \subseteq B \times C \) if and only if \( A \subseteq B \).

Have to show \((\Rightarrow), (\Leftarrow)\).

\((\Rightarrow)\): Assume \( A \times C \subseteq B \times C \).

\[ \forall (x,y) \in A \times C \text{ then } (x,y) \in B \times C. \]

Need to show \( \forall x \in A \text{ let } c \in C \), \( c \neq \emptyset \).

so that \( (a,c) \in A \times C \implies (a,c) \in B \times C \)

\[ \therefore A \subseteq B. \]

\((\Leftarrow)\): Assume \( A \subseteq B \).

\[ \forall (x,y) \in A \times C \text{ then } \exists x \in A \land y \in C \text{ by assumption} \]

So \( x \in B \land y \in C \)

\[ \therefore (x,y) \in B \times C \]

\[ \therefore A \times C \subseteq B \times C. \]

Note: \( \forall x \in A \subseteq B \), then \( A \times C \subseteq B \times C \).

\[ \text{But } A \times C \subseteq B \times C \Rightarrow A \subseteq B \text{ if } C \neq \emptyset \]

Example. \( A = \{1,2\}, \ C = \emptyset \)

\[ B = \{3,4\} \]

\[ \text{But } A \times C = \emptyset \]

\[ B \times C = \emptyset \]

\[ A \times C \neq B \times C \]

\[ A \times C \subseteq B \times C \]
§3.3. Proof by contrapositive To prove "If \( P(x) \) then \( Q(x) \)." we prove "If \( \neg Q(x) \), then \( \neg P(x) \)."

Examples: If \( n \in \mathbb{Z} \) is such that \( 15n \) is even, then \( 3n \) is even.

An integer \( n \) is odd (even) if and only if \( n^2 \) is odd (even).

**Proof**

\[ \text{Let } n \in \mathbb{Z} \text{. Prove } n \text{ is odd if and only if } n^2 \text{ is odd.} \]

\[ \text{(\( \Rightarrow \)) Assume } n \text{ is odd, i.e., } n = 2l + 1, \quad l \in \mathbb{Z} \]

Then \( n^2 = (2l + 1)^2 = 4l^2 + 4l + 1 = 2(2l^2 + 2l) + 1 \)

\[ \therefore \quad n^2 = 2m + 1 \quad \text{with } m = l^2 + l \in \mathbb{Z} \]

\[ \text{(\( \Leftarrow \)) Assume } n^2 \text{ is odd. Need to prove } n \text{ is odd.} \]

Assume \( n^2 = 2m + 1, m \in \mathbb{Z} \). Need to prove \( n = 2l + 1, l \in \mathbb{Z} \)

We prove by using the contrapositive of the statement:

Assume \( n \in \mathbb{Z}, n \) is even

\[ n = 2l, \quad l \in \mathbb{Z} \]

\[ \therefore \quad n^2 = 4l^2 = 2(2l^2) \quad m = 2l^2 \in \mathbb{Z} \]

\[ \therefore \quad n^2 \text{ is even} \]

(\( P \Rightarrow Q \) \( \equiv (\neg Q \Rightarrow \neg P) \))
§5.2 Proof by contradiction To prove \( P \implies Q \), show that \( P \land \neg Q \) is impossible.

Also, to prove \( P \), assume \( \neg P \) and derive a contradiction.

Examples The sum of a rational number and an irrational number is irrational.
The number \( \sqrt{2} \) is irrational. (If \( x = \sqrt{2} \), then \( x \) is irrational.)
There are infinitely many prime numbers.
(If \( S \) is the set of primes, then \( S \) has infinitely many elements.)
If \( x, y \in \mathbb{R} \) are positive, then \( \sqrt{x} + \sqrt{y} \neq \sqrt{x + y} \).

\[
\begin{array}{c|c|c|c}
T & F & T & F \\
\hline
T & T & F & T \\
\end{array}
\]

**Proof by contradiction**

Assume \( x = \sqrt{2} \in \mathbb{R} \), then \( x \) is irrational.

Assume \( x = \sqrt{2} \).

\[ x = \sqrt{2} \neq \frac{m}{n}, \quad m, n \in \mathbb{Z}. \]

\[ \therefore x \text{ is irrational} \]

**Indirect proof:** Assume \( x \) is rational

\[ \text{Then} \quad x \neq \sqrt{2} \]

We may assume \( m, n \) have no common factors.

Then \[ 2 = \frac{m^2}{n^2} \quad \text{if} \quad 2n^2 = m^2 \]

So \( n^2 \) is even and so is \( m \). Thus \( m = 2k \) and

\[ 2n^2 = m^2 = (2k)^2 = 4k^2. \]

\[ \therefore n^2 = 2k^2 \text{ is even} \]

Thus \( m \) \& \( n \) have a common factor.
Example

There are infinitely many prime numbers.

If $S$ is the set of prime numbers, then $S$ is infinite.

Proof:
Assume $S$ is the set of prime numbers and assume the contrary that $S$ is finite.

So $S = \{p_1, p_2, \ldots, p_n\}$.

Consider $q = p_1 p_2 \cdots p_n + 1$.

Case 1: $q$ is a prime number.

Then $q \notin S$, which is a contradiction.

Case 2: $q$ is not a prime number.

Then $q$ has a prime factor $p$.

and $q \neq p_i$ for any $i$.

Because $p_i$ is not a factor of $q$, as $q = p_i \left\lfloor \frac{m_i}{p_i} \right\rfloor + 1$

So $p$ is a prime number not in $S$.

which is a contradiction.
Example

The sum of a rational number and an irrational number is irrational.

Proof: Reformulate:

If \( x = \frac{m}{n}, \ m, n \in \mathbb{Z}, \ n \neq 0 \)

and \( y \neq \frac{r}{s} \) for any \( a, b \in \mathbb{Z}, \ b \neq 0 \).

Then \( x + y \neq \frac{r}{s} \) for any \( r, s \in \mathbb{Z}, \ s \neq 0 \).

Proof: Assume \( x = \frac{m}{n}, \ m, n \in \mathbb{Z}, \ n \neq 0 \)

Assume \( y \neq \frac{r}{s} \) for any \( a, b \in \mathbb{Z}, \ b \neq 0 \).

Assume \( x + y = \frac{r}{s} \) for any \( r, s \in \mathbb{Z}, \ s \neq 0 \).

Then \( y = (x + y) - x = \frac{r}{s} - \frac{m}{n} = \frac{rn - ms}{sn} = \frac{p}{q} \)  \( p = rn - ms \in \mathbb{Z} \)

\( q = sn \neq 0 \)

\( \Rightarrow \ y \in \mathbb{Q} \)

\( \Rightarrow \) a contradiction

Example:

If \( x, y \in \mathbb{R}, \ x, y > 0 \), then \( x + y > \sqrt{x+y} \).

Proof:

Assume \( x, y \in \mathbb{R}, \ x, y > 0 \).

Assume the contrary that \( \sqrt{x+y} = \sqrt{x} + \sqrt{y} \).

Then \( (\sqrt{x+y})^2 = (\sqrt{x} + \sqrt{y})^2 \)

\( x + 2\sqrt{xy} + y = x + y \)

\( \Rightarrow \ 2\sqrt{xy} = 0 \)

\( \Rightarrow \ x = 0, y = 0 \) a contradiction.
Hint on Homework:

# 8

\[ S = \{ n \in \mathbb{N} : \sqrt{n} \text{ is irrational} \} \]

\[ A = \{ m^2 : m \in \mathbb{N} \} \subseteq S \]

Reason: \( x \in A, \ x = m^2 \)

So that \( \sqrt{x} = \sqrt{m^2} \) is irrational.

To prove:

\( S \) has no maximum number.

Assume \( N \in S \) is max.

i.e., \( \forall x \in S \text{ then } x \leq N \).

Hint. Then consider \( 2N^2 \).